

BORDERED FLOER HOMOLOGY AND THE SPECTRAL SEQUENCE OF A BRANCHED DOUBLE COVER II: THE SPECTRAL SEQUENCES AGREE

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ABSTRACT. Given a link in the three-sphere, Ozsváth and Szabó showed that there is a spectral sequence starting at the Khovanov homology of the link and converging to the Heegaard Floer homology of its branched double cover. The aim of this paper is to explicitly calculate this spectral sequence in terms of bordered Floer homology. There are two primary ingredients in this computation: an explicit calculation of bimodules associated to Dehn twists, and a general pairing theorem for polygons. The previous part [LOT14a] focuses on computing the bimodules; this part focuses on the pairing theorem for polygons, in order to prove that the spectral sequence constructed in the previous part agrees with the one constructed by Ozsváth and Szabó.

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1. INTRODUCTION

This paper concerns the relationship between Khovanov homology for links L in the three-sphere and the Heegaard Floer homology groups of their branched double covers. These two link invariants are related, as was shown in joint work of Zoltán Szabó and the second author:

Theorem 1. [[OSz05](#), Theorem 1.1] *For any link $L \subset S^3$ there is a spectral sequence with E_2 -page the reduced Khovanov homology of the mirror of L and E_∞ -page $\widehat{HF}(\Sigma(L))$.*

The differentials in the spectral sequence come from counts of pseudo-holomorphic polygons. As such, they depend on a number of choices. Baldwin showed [[Bal11](#)] that the filtered chain homotopy type inducing the spectral sequence (and hence each page after E_2) is a link invariant, see also [[Rob08](#)].

The present work is a sequel to [[LOT14a](#)]. In that paper, we gave a combinatorially-defined spectral sequence from the reduced Khovanov homology $\widehat{Kh}(m(L))$ of the mirror of a link L to the Heegaard Floer homology $\widehat{HF}(\Sigma(L))$ of the double cover of S^3 branched over L . That spectral sequence was *a priori* different from the one in [[OSz05](#)]. The spectral sequence from [[LOT14a](#)] arises from decomposing the branched double cover of a link into a union of bordered manifolds as in [[LOT08](#), [LOT15](#)]. Bordered Floer homology associates a bimodule to (the branched double cover of) each link crossing. The bimodule associated to a crossing can be interpreted as a mapping cone of bimodules corresponding to the (branched double covers of the) two resolutions:

Proposition 1.1. [[LOT14a](#), Theorem 2] *Consider the tangle consisting of k horizontal segments lying in a plane. We think of this as a tangle in $S^2 \times I$. Its branched double cover is a bordered three-manifold representing $[0, 1] \times \Sigma$. We denote this bordered three-manifold by \mathbb{I} (as it represents the identity cobordism). Next, consider the tangle σ_i (respectively σ_i^{-1}) consisting of horizontal segments with exactly one positive (respectively negative) crossing, which occurs between the i^{th} and $(i+1)^{\text{st}}$ strands. Let $\check{\sigma}_i$ denote the “anti-braidlike” resolution of σ_i and let $\widehat{CFDA}(\sigma_i)$ and $\widehat{CFDA}(\check{\sigma}_i)$ denote the bimodules associated to the branched double covers of σ_i and $\check{\sigma}_i$, respectively (which are bordered 3-manifolds). Then there are*

$$\begin{array}{ccc}
\begin{array}{c} \text{---} \\ \text{---} \\ \diagdown \quad \diagup \\ \text{---} \\ \text{---} \\ \sigma_i^{-1} \end{array} & = \text{Cone} \left(\begin{array}{ccc} \text{---} & & \text{---} \\ \text{---} & \xrightarrow{F^-} & \text{---} \\ \text{---} & & \text{---} \\ \text{---} & & \text{---} \\ \hat{\sigma}_i & & \check{\sigma}_i \end{array} \right) & \quad \quad \quad \begin{array}{ccc} \text{---} & & \text{---} \\ \text{---} & \xleftarrow{F^+} & \text{---} \\ \text{---} & & \text{---} \\ \text{---} & & \text{---} \\ \hat{\sigma}_i & & \check{\sigma}_i \end{array} \\
\end{array}$$

FIGURE 1. **Braid generators as mapping cones.**

distinguished bimodule morphisms

$$\begin{aligned}
F^- : \widehat{CFDA}(\mathbb{I}) &\rightarrow \widehat{CFDA}(\check{\sigma}_i) \\
F^+ : \widehat{CFDA}(\check{\sigma}_i) &\rightarrow \widehat{CFDA}(\mathbb{I})
\end{aligned}$$

with the property that $\widehat{CFDA}(\sigma_i^{-1})$ and $\widehat{CFDA}(\sigma_i)$ are isomorphic to the mapping cones of the bimodule morphisms F^- and F^+ respectively. (See Figure 1.)

Remark 1.2. If we orient the tangle locally so that the strands are all oriented from left to right then the sign of the crossing appearing in the above proposition coincides with the sign of the crossing from knot theory.

Next, the pairing theorem for bordered Floer homology realizes the Heegaard Floer complex of the branched double cover as a tensor product of the constituent bimodules. Since each of these is a mapping cone, the iterated tensor product inherits a filtration by $\{0, 1\}^n$. The homology of the associated graded complex can be identified with the chain groups computing reduced Khovanov homology; see Corollary 5.41, below. Thus, one might expect that the spectral sequence associated to the filtration is related to the spectral sequence from Theorem 1.

Indeed, the aim of the present paper is to identify these two spectral sequences; i.e. to prove:

Theorem 2. *The spectral sequence coming from [LOT14a, Theorem 3] agrees with the spectral sequence for multi-diagrams from [OSz05, Theorem 1.1] (Theorem 1 above). In fact, the filtered chain complexes inducing these two spectral sequences are filtered chain homotopy equivalent.*

Our strategy of proof involves the following four key steps:

- (1) Generalize the polygon counts which appear in Heegaard Floer homology (or more generally, in Fukaya categories) to the bordered context.
- (2) Give a pairing theorem which identifies polygon counts in a closed diagram in terms of pairings of polygon counts appearing in the bordered context.
- (3) Construct a triple diagram corresponding to each crossing, and identify the corresponding triangle-counting morphism with the DA bimodule morphism appearing in Proposition 1.1.
- (4) Draw a Heegaard multi-diagram for the branched double cover with the property that the polygon counts in the diagram decompose as pairings, in the sense of Step 2, of the triangles appearing in Step 3.

In the above outline, Step 1 is a routine. Step 3 will follow quickly from a uniqueness property of the maps appearing in Proposition 1.1, which was established in [LOT14a]. Step 4 is in fact the diagram used in [LOT14a], which in turn is a stabilization of a diagram studied by Greene [Gre13].

Thus, the crux of the present paper is the pairing theorem in Step 2. To this end, it is important to formulate a well-defined object: holomorphic polygon counts depend on the precise conformal parameters of the multi-diagram, while pairing theorems (such as the pairing theorem for closed manifolds, [LOT08, Theorem 1.3]) depend on degenerating these parameters. Thus, we need an object depending on holomorphic polygon counts which is sufficiently robust to be invariant under variation of parameters. This object is provided by the notion of a \mathbb{I} -filtered chain complex of attaching circles, which consists of a collection of sets of attaching circles $\{\beta^i\}$ together with a suitable collection of chains $\eta^{i_1 < i_2} \in \widehat{CF}(\beta^{i_1}, \beta^{i_2}, z)$ (Definition 3.14). (This is a special case of a notion of twisted complexes which play a prominent role in the theory of Fukaya categories [Sei08]; see also Remark 3.16.) The key point is that this data makes $\bigoplus_{i \in \mathbb{I}} \widehat{CF}(\alpha, \beta^i, z)$ into a filtered chain complex (Definition 3.18):

Proposition 1.3. *Given a finite partially ordered set \mathbb{I} , an \mathbb{I} -filtered chain complex of attaching circles $\{\beta^i\}_{i \in \mathbb{I}}$, and one more set of attaching circles α , there is a naturally associated \mathbb{I} -filtered complex denoted $\widehat{CF}(\alpha, \{\beta^i\}_{i \in \mathbb{I}}, z)$ whose associated graded complex is*

$$\bigoplus_{i \in \mathbb{I}} \widehat{CF}(\alpha, \beta^i, z).$$

(This is reformulated and proved as Proposition 3.19, below.)

Now, the spectral sequence from Theorem 1 can be thought of as coming from a gluing statement for complexes of attaching circles, as follows:

Proposition 1.4. *Fix surfaces-with-boundary Σ_1 and Σ_2 with common boundary C , and let $\{\beta^i\}_{i \in \mathbb{I}}$ be a chain complex of attaching circles in Σ_1 , filtered by a partially ordered set \mathbb{I} , and let $\{\gamma^j\}_{j \in \mathbb{J}}$ be a chain complex of attaching circles in Σ_2 filtered by a partially ordered set \mathbb{J} . Then these chain complexes of attaching circles can be glued to form a natural $\mathbb{I} \times \mathbb{J}$ -filtered chain complex of attaching circles $\beta \# \gamma$ in $\Sigma_1 \cup_C \Sigma_2$.*

The construction of the $\mathbb{I} \times \mathbb{J}$ -filtered chain complex is given in Definition 3.40 below; a more precise version of Proposition 1.4 is given as Proposition 3.52.

Simplified versions of Step 1 are provided by the following two bordered analogues of Proposition 1.3.

Theorem 3. *Let Σ be a surface-with-boundary and $\{\beta^i\}_{i \in \mathbb{I}}$ an \mathbb{I} -filtered chain complex of attaching circles in Σ . Let α be a collection of α -arcs and α -circles so that each $(\Sigma, \alpha, \beta^i)$ is a bordered Heegaard diagram. Then there is a naturally associated \mathbb{I} -filtered A_∞ -module $\widehat{CFA}(\alpha, \{\beta^i\}_{i \in \mathbb{I}}, z)$ (respectively \mathbb{I} -filtered type D structure $\widehat{CFD}(\alpha, \{\beta^i\}_{i \in \mathbb{I}}, z)$) whose associated graded object is $\bigoplus_{i \in \mathbb{I}} \widehat{CFA}(\Sigma, \alpha, \beta^i, z)$ (respectively $\bigoplus_{i \in \mathbb{I}} \widehat{CFD}(\Sigma, \alpha, \beta^i, z)$).*

(This is proved as Propositions 4.27 and 4.29.)

The tensor product over the bordered algebra $\mathcal{A}(\mathcal{Z})$ of an \mathbb{I} -filtered A_∞ -module and a \mathbb{J} -filtered type D structure is naturally an $\mathbb{I} \times \mathbb{J}$ -filtered chain complex; see for instance [LOT14a, Section 2] and Lemma 2.9 below. A simplified version of the pairing theorem for polygons needed in Step 2 can be stated as follows.

Theorem 4. *Let Σ_1 and Σ_2 be surfaces, and $\alpha^i \subset \Sigma_i$ be α -curves which intersect the boundary of Σ_i in a pointed matched circle $\mathcal{Z}_i = \partial\Sigma_i \cap \alpha_i$, with $\mathcal{Z}_1 = -\mathcal{Z}_2$. Let $\alpha = \alpha_1 \cup \alpha_2 \subset \Sigma = \Sigma_1 \cup_{\partial} \Sigma_2$. Fix an \mathbb{I} -filtered chain complex of attaching circles $\{\beta^i\}_{i \in \mathbb{I}}$ in Σ_1 and a \mathbb{J} -filtered chain complex of attaching circles $\{\gamma^j\}_{j \in \mathbb{J}}$ in Σ_2 . We can form the $\mathbb{I} \times \mathbb{J}$ -filtered chain complex of attaching circles $\beta \# \gamma$, as in Proposition 1.4. Then there is a homotopy equivalence of $\mathbb{I} \times \mathbb{J}$ -filtered chain complexes*

$$\widehat{\text{CF}}(\alpha_1 \cup \alpha_2, \{\beta^i \# \gamma^j\}_{(i,j) \in \mathbb{I} \times \mathbb{J}}, z) \simeq \widehat{\text{CFA}}(\alpha_1, \{\beta^i\}_{i \in \mathbb{I}}) \boxtimes \widehat{\text{CFD}}(\alpha_2, \{\gamma^j\}_{j \in \mathbb{J}}).$$

(This is proved as Theorem 5, in Section 5.)

We prove Theorem 4 by applying “time dilation”, the deformation used in [LOT08] to prove the pairing theorem for three-manifolds. As a preliminary step, though, we will need to translate time, as in the proof of the self-pairing theorem [LOT15, Theorem 7].

The generalizations of Theorems 3 and 4 to the bimodule case (Theorem 7 below), which we need in Step 2, is a fairly routine extension.

1.1. Organization. In Section 2, we remind the reader of the definition of the \mathbb{I} -filtered algebraic objects which will appear throughout: complexes, A_∞ -modules, type D structures, and bimodules. We also explain how the tensor products of filtered objects give filtered complexes. Section 3 concerns pseudo-holomorphic polygons in Heegaard multi-diagrams. In that section, we explain the definition of chain complexes of attaching circles, showing how these can be used to construct filtered chain complexes (Proposition 1.3), and how they can be glued (Proposition 1.4). In Section 4, the constructions are generalized to bordered multi-diagrams. That section contains the proof of Theorem 3 (restated as Propositions 4.27 and 4.29 in the type A and D cases, respectively). Proposition 4.36 is an analogous result for bimodules.

Having described all the ingredients in the statement of the pairing theorem for polygons, in Section 5 we turn to the precise statement and proof of Theorem 4. (Theorem 5 is the simpler, module version and Theorem 7 is the more general, bimodule version.) As a consequence of the pairing theorem, we deduce that the surgery exact sequence implied by bordered Floer homology [LOT08, Section 11.2] agrees with the original surgery sequence from [OSz04a]. In Section 6, we describe the Heegaard multi-diagram for double covers, and show how it can be used, in conjunction with the pairing theorem, to prove Theorem 2.

1.2. Further remarks. The present paper is devoted to computing a spectral sequence from Khovanov homology to the Heegaard Floer homology of a branched double cover. It is worth pointing out that this spectral sequence has a number of generalizations to other contexts. For example, Grigsby and Wehrli have established an analogous spectral sequence starting at various colored versions of Khovanov homology and converging to knot homology of the branch locus in various branched covers of L , leading to a proof that these colored Khovanov homology groups detect the unknot [GW10]. Bloom [Blo11] proved an analogue of Theorem 1 using Seiberg-Witten theory in place of Heegaard Floer homology. More recently, Kronheimer and Mrowka [KM07] have proved an analogue with \mathbb{Z} coefficients, converging to a version of instanton link homology, showing that Khovanov homology detects the unknot.

We have relied in this paper extensively on the machinery of bordered Floer homology, which gives a technique for computing \widehat{HF} for three-manifolds. Another powerful technique for computing this invariant is the technique of *nice diagrams*; see [SW10]. Indeed, although nice diagrams were originally conceived as a way to compute \widehat{HF} for closed three-manifolds,

it has been extended to a tool for computing triangle maps in [LMW08], see also [Sar11]. At present, a “nice” technique for directly computing the polygons needed in Theorem 1 has not been developed. See [MO10, Theorem 11.10] for an alternative approach studying these maps.

Ideas from [Aur10] suggest another route to the pairing theorem for polygons.

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2. ALGEBRAIC PRELIMINARIES

In this section we recall some notions about filtered objects, partly to fix notation. Much of the material overlaps with [LOT14a, Section 2]; we have also included here some of the discussion from [LOT15, Section 2].

We use Mor to denote the chain complex of morphisms in a dg category. For example, in the dg category of chain complexes (over a ground ring \mathbf{k} of characteristic 2), $\text{Mor}(C_*, D_*)$ denotes the \mathbf{k} -module maps from C_* to D_* , with differential given by $d(f) = f \circ \partial_C + \partial_D \circ f$. So, for instance, the cycles in $\text{Mor}(C_*, D_*)$ are the chain maps. For more details on the categories we work with, see [LOT15, Section 2].

Definition 2.1. *Let \mathbb{I} be a finite, partially ordered set. An \mathbb{I} -filtered chain complex is a collection $\{C^i\}_{i \in \mathbb{I}}$ of chain complexes over \mathbb{F}_2 together with a collection of degree-zero morphisms $D^{i < j} \in \text{Mor}(C^i, C^j[1])$ for each $i, j \in I$ with $i < j$, satisfying the compatibility condition*

$$d(D^{i < k}) = \sum_{\{j | i < j < k\}} D^{j < k} \circ D^{i < j},$$

where d denotes the differential on the morphism space $\text{Mor}(C^i, C^k[1])$.

(Here and elsewhere, $[1]$ denotes a grading shift by 1. In the rest of this paper, the modules and chain complexes considered will be ungraded, but we will keep track of gradings in this background section.)

In particular, the compatibility condition implies that if i and j are consecutive, then $D^{i < j}$ is a chain map.

This can be reformulated in the following more familiar terms. Given $\{C^i, D^{i < j}\}$, we can form the graded vector space $C = \bigoplus_{i \in \mathbb{I}} C^i$, equipped with the degree -1 endomorphism $D: C \rightarrow C$ defined by $D = \sum_i \partial^i + \sum_{i < j} D^{i < j}$, where ∂^i is the differential on C^i . The compatibility condition is simply the statement that D is a differential. The pair (C, D) is naturally a filtered complex. The associated graded complex to C is simply $\bigoplus_{i \in \mathbb{I}} C^i$, equipped with the differential ∂ which is the sum of the differentials on the C^i .

This notion has several natural generalizations to the case of A_∞ -modules. The one we will use is the following. (See also Remark 2.13.) First, fix an A_∞ -algebra \mathcal{A} over a ground ring \mathbf{k} of characteristic 2, and assume that \mathcal{A} is *bounded* in the sense that the operations μ_i on \mathcal{A} vanish identically for all sufficiently large i . Recall that the category of A_∞ -modules over \mathcal{A} is a dg category [LOT15, Section 2.2.2].

Definition 2.2. Let \mathbb{I} be a finite, partially ordered set. An \mathbb{I} -filtered A_∞ -module over \mathcal{A} is a collection $\{M^i\}_{i \in \mathbb{I}}$ of A_∞ -modules over \mathcal{A} , equipped with a preferred morphism $F^{i < j} \in \text{Mor}(M^i, M^j[1])$ for each pair $i, j \in I$ with $i < j$, satisfying the compatibility condition

$$(2.3) \quad d(F^{i < k}) = \sum_{j | i < j < k} F^{j < k} \circ F^{i < j},$$

where d denotes the differential on the morphism space $\text{Mor}(M^i, M^k[1])$. Moreover, we say that the filtered module is bounded if for all sufficiently large n , the maps

$$m_n: M^i \otimes \overbrace{\mathcal{A} \otimes \cdots \otimes \mathcal{A}}^{n-1} \rightarrow M^i[2-n] \quad \text{and} \quad F_n^{i < k}: M^i \otimes \overbrace{\mathcal{A} \otimes \cdots \otimes \mathcal{A}}^{n-1} \rightarrow M^k[2-n]$$

vanish.

We can form the graded \mathbf{k} -module $M = \bigoplus_{i \in \mathbb{I}} M^i$. The higher products m_n on the M^i and the maps $F^{i < j}$ can be assembled to give a map $M \otimes \mathcal{T}^*(\mathcal{A}[1]) \rightarrow M[1]$ defined, for $x_i \in M^i$ and $a_1, \dots, a_n \in \mathcal{A}$, by

$$(2.4) \quad x_i \otimes a_1 \otimes \cdots \otimes a_n \mapsto m_{n+1}^i(x_i, a_1, \dots, a_n) + \sum_{i < j} F_{n+1}^{i < j}(x_i, a_1, \dots, a_n),$$

where m^i denotes the A_∞ action on M^i . The compatibility condition on the $F^{i < j}$ is equivalent to Formula (2.4) defining an A_∞ -module structure. The boundedness condition ensures that the above constructions define a map $M \otimes \overline{\mathcal{T}}^*(\mathcal{A}[1]) \rightarrow M[1]$, replacing the tensor algebra $\mathcal{T}^*(\mathcal{A}) = \bigoplus_{i \geq 0} \mathcal{A}^{\otimes i}$ by its completion $\overline{\mathcal{T}}^*(\mathcal{A}) = \prod_{i \geq 0} \mathcal{A}^{\otimes i}$.

Recall that a type D structure (a variant of twisted complexes) over the A_∞ -algebra \mathcal{A} over $\mathbf{k} = \bigoplus_{i=1}^N \mathbb{F}_2$ is a \mathbf{k} -module X together with a \mathbf{k} -linear map $\delta^1: X \rightarrow \mathcal{A}[1] \otimes_{\mathbf{k}} X$ so that

$$\begin{array}{c} \downarrow \\ \delta^1 \\ \swarrow \quad \downarrow \\ \mu_1 \quad \downarrow \\ \downarrow \end{array} + \begin{array}{c} \downarrow \\ \delta^1 \\ \swarrow \quad \downarrow \\ \mu_2 \quad \downarrow \\ \downarrow \end{array} + \begin{array}{c} \downarrow \\ \delta^1 \\ \swarrow \quad \downarrow \\ \mu_3 \quad \downarrow \\ \downarrow \end{array} + \cdots = 0.$$

(See [LOT15, Definition 2.2.23].) Over a dg algebra only the first two terms occur. If X and Y are type D structures, the morphism space $\text{Mor}^D(X, Y)$ is the space of \mathbf{k} -linear maps $h^1: X \rightarrow \mathcal{A} \otimes Y$, equipped with a differential consisting of terms each of which applies δ_X^1 some number of times, then h^1 , and then finally δ_Y^1 a number of times, and then feeds all the n resulting algebra elements into the operation μ_n on \mathcal{A} . When \mathcal{A} is a dg algebra, this

differential can be written

$$d(h^1) = \begin{array}{c} \downarrow \\ h^1 \\ \swarrow \downarrow \\ \mu_1 \downarrow \end{array} + \begin{array}{c} \downarrow \delta_X^1 \\ \downarrow h^1 \\ \swarrow \downarrow \\ \mu_2 \downarrow \end{array} + \begin{array}{c} \downarrow \\ h^1 \\ \downarrow \delta_Y^1 \\ \swarrow \downarrow \\ \mu_2 \downarrow \end{array} .$$

Suppose for simplicity that \mathcal{A} is a dg algebra. Given $g^1 \in \text{Mor}^{\mathcal{A}}(X, Y)$ and $h^1 \in \text{Mor}^{\mathcal{A}}(Y, Z)$, we can define their composite $(h^1 \circ g^1)$ by the expression

$$\begin{array}{c} \downarrow \\ g^1 \\ \swarrow \downarrow \\ \mu_2 \downarrow \end{array} \begin{array}{c} \downarrow \\ h^1 \\ \downarrow \end{array} .$$

We define a filtered type D structure similarly to a filtered A_∞ -module:

Definition 2.5. Fix a dg algebra \mathcal{A} , and let \mathbb{I} be a finite, partially ordered set. An \mathbb{I} -filtered type D structure over \mathcal{A} is a collection $\{P^i\}_{i \in \mathbb{I}}$ of type D structures over \mathcal{A} , equipped with preferred degree-zero morphisms $h_{i < j}^1: P^i \rightarrow \mathcal{A}[1] \otimes P^j$ (for each pair $i, j \in \mathbb{I}$ with $i < j$) satisfying the compatibility condition (2.3).

Remark 2.6. Over an A_∞ -algebra, type D structures form an A_∞ -category [LOT15, Lemma 2.2.27], making this definition more complicated. We will not need that level of generality in our present considerations.

Let $M = (\{M_i\}_{i \in \mathbb{I}}, \{F^{i < i'}\}_{i < i' \in \mathbb{I}})$ be an \mathbb{I} -filtered A_∞ -module over \mathcal{A} and let $P = (\{P^j\}_{j \in \mathbb{J}}, \{h_{j < j'}^1\}_{j < j' \in \mathbb{J}})$ be a \mathbb{J} -filtered type D structure over \mathcal{A} . Suppose moreover that M is bounded. We can form the tensor product

$$M \boxtimes P = \bigoplus_{i \times j \in \mathbb{I} \times \mathbb{J}} M^i \otimes P^j,$$

endowed with a differential

$$\partial: M^i \otimes P^j \rightarrow M^i \otimes P^j[1]$$

given by

$$\partial = \begin{array}{c} \downarrow \quad \searrow \\ m^i \quad \delta_j \\ \downarrow \quad \downarrow \end{array},$$

where $\delta_j: P^j \rightarrow \overline{\mathcal{T}}^*(\mathcal{A}[1]) \otimes P^j$ is the map obtained by iterating δ^1 on P^j and

$$m^i: M_i \otimes \overline{\mathcal{T}}^*(\mathcal{A}[1]) \rightarrow M_i[1]$$

is the map induced by the A_∞ -action. Define maps $D^{i \times j < i' \times j'}$ by the following expression:

$$(2.7) \quad D^{i \times j < i' \times j'} = \sum_{n=0}^{\infty} \sum_{j=j_0 < \dots < j_n=j'} \begin{array}{c} \downarrow \quad \delta_{j_0} \\ \downarrow \quad \delta_{j_0} \\ h_{j_0 < j_1}^1 \\ \downarrow \quad \delta_{j_1} \\ \vdots \\ \downarrow \quad \delta_{j_{n-1}} \\ h_{j_{n-1} < j_n}^1 \\ \downarrow \quad \delta_{j_n} \\ F^{i \leq i'} \end{array},$$

with the understanding that

$$F^{i \leq i'} = \begin{cases} F^{i < i'} & \text{if } i < i' \\ m^i & \text{if } i = i' \end{cases}$$

We can formulate a boundedness condition on filtered type D structures analogous to the one for type A structures:

Definition 2.8. *The filtered type D structure on P is called bounded if for each $j \in \mathbb{J}$, sufficiently large iterates of δ^1 on P_j vanish, i.e., the map δ_j maps to $\mathcal{T}^*(\mathcal{A}[1]) \otimes P^j \subset \overline{\mathcal{T}}^*(\mathcal{A}[1]) \otimes P^j$.*

If M is not bounded, but P is, the tensor product $M \boxtimes P$ still makes sense. In fact, the following is part of [LOT14a, Lemma 2.4].

Lemma 2.9. *Let M be an \mathbb{I} -filtered A_∞ -module over \mathcal{A} and let P be a \mathbb{J} -filtered type D structure over \mathcal{A} . Assume that one of M or P is bounded, so $M \boxtimes P$ is defined. Then, $M \boxtimes P$ is naturally an $\mathbb{I} \times \mathbb{J}$ -filtered chain complex.*

2.1. Bimodules. There are analogous constructions of filtered bimodules, subsuming both filtered A_∞ -modules and filtered type D structures. Fix A_∞ -algebras \mathcal{A} and \mathcal{B} over ground rings $\mathbf{k} = \bigoplus_{i=1}^N \mathbb{F}_2$ and $\mathbf{l} = \bigoplus_{i=1}^{N'} \mathbb{F}_2$, respectively. Recall that a *type DA bimodule* over \mathcal{A} and \mathcal{B} consists of a (\mathbf{k}, \mathbf{l}) -bimodule X together with a map $\delta^1: X \otimes_{\mathbf{l}} T^*(\mathcal{B}[1]) \rightarrow (\mathcal{A} \otimes_{\mathbf{k}} X)[1]$ satisfying:

$$\begin{array}{c} \downarrow \swarrow \\ \delta^1 \\ \swarrow \downarrow \\ \mu_1 \downarrow \end{array} + \begin{array}{c} \downarrow \swarrow \\ \delta^1 \\ \downarrow \swarrow \delta^1 \\ \mu_2 \downarrow \end{array} + \begin{array}{c} \downarrow \swarrow \\ \delta^1 \\ \downarrow \swarrow \delta^1 \\ \downarrow \swarrow \delta^1 \\ \mu_3 \downarrow \end{array} + \cdots = 0.$$

See [LOT15, Definition 2.2.43]. The space of morphisms between type DA bimodules X and Y is the graded space of maps $h_k^1: X \otimes \mathcal{T}^*(\mathcal{B}[1]) \rightarrow \mathcal{A} \otimes Y$. Restricting to the case that \mathcal{A} is a dg algebra, this morphism space has differential

$$d(h^1) = \begin{array}{c} \downarrow \swarrow \\ h^1 \\ \swarrow \downarrow \\ \mu_1 \downarrow \end{array} + \begin{array}{c} \downarrow \swarrow \\ \delta^1 \\ \downarrow \swarrow \delta^1 \\ h^1 \\ \swarrow \downarrow \\ \mu_2 \downarrow \end{array} + \begin{array}{c} \downarrow \swarrow \\ h^1 \\ \downarrow \swarrow \delta^1 \\ \delta^1 \\ \swarrow \downarrow \\ \mu_2 \downarrow \end{array}.$$

If $h^1 \in \text{Mor}(M, N)$ and $g^1 \in \text{Mor}(N, P)$ are morphisms of DA bimodules over \mathcal{A} and \mathcal{B} , and \mathcal{A} is a dg algebra, we can define their composite by the expression:

$$(g^1 \circ h^1) = \begin{array}{c} \downarrow \swarrow \\ h^1 \\ \downarrow \swarrow \\ g^1 \\ \swarrow \downarrow \\ \mu_2 \downarrow \end{array}$$

Boundedness for type DA bimodules is slightly subtle to formulate. Before doing so, we need some more notation. First, we will assume that our DA bimodule is defined over \mathcal{A} and \mathcal{B} , where \mathcal{A} and \mathcal{B} are augmented over their ground rings, with augmentation $\epsilon: \mathcal{A} \rightarrow \mathbf{k}$ and $\epsilon: \mathcal{B} \rightarrow \mathbf{l}$ and augmentation ideals \mathcal{A}_+ and \mathcal{B}_+ .

For each partition of $(1, \dots, i)$ into m subsequences and each length $m - j$ subsequence of $(1, \dots, m)$, we have a corresponding map $M \otimes \mathcal{B}_+[1]^{\otimes i} \rightarrow \mathcal{A}[1]^{\otimes j} \otimes M$, defined by applying δ^1 m times with input $x \in M$ and further algebra inputs specified by the partition, followed by the map $\mathcal{A}[1]^{\otimes m} \rightarrow \mathcal{A}[1]^{\otimes j}$ obtained by applying the augmentation map ϵ to the tensor factors in the length $m - j$ subsequence. Such a map is called a *spinal DA bimodule operation* with i algebra inputs and j outputs.

Definition 2.10. [LOT15, Definition 2.2.46] *The filtered type DA bimodule M is called operationally bounded if for each x there is an n so that all spinal DA bimodule operations with module input x , i algebra inputs, and j algebra outputs, with $i + j > n$, vanish. It is called left bounded if for each $x \in M$ and each i , there is an n so that all spinal DA bimodule operations with module input x , i algebra inputs, and $j > n$ algebra outputs vanish. It is called right bounded if for each $x \in M$ and each j , there is an n so that all spinal DA bimodule operations with module input x , $i > n$ algebra inputs, and j algebra outputs vanish.*

We define filtered type DA structures similarly to filtered A_∞ -modules and type D structures:

Definition 2.11. *Fix dg algebras \mathcal{A} and \mathcal{B} , and let \mathbb{I} be a finite, partially ordered set. An \mathbb{I} -filtered type DA structure over \mathcal{A} and \mathcal{B} is a collection $\{Q^i\}_{i \in \mathbb{I}}$ of type DA structures over \mathcal{A} and \mathcal{B} , equipped with a preferred degree-zero morphism $F^{i < j} \in \text{Mor}(P^i, P^j[1])$ for each pair $i, j \in \mathbb{I}$ with $i < j$, satisfying the compatibility condition (2.3).*

The following is the bimodule analogue of Lemma 2.9 (which is also [LOT14a, Lemma 2.7]):

Lemma 2.12. *Let \mathcal{A} , \mathcal{B} , and \mathcal{C} be dg algebras. Let ${}^{\mathcal{A}}M_{\mathcal{B}}$ be an \mathbb{I} -filtered DA-bimodule and ${}^{\mathcal{B}}N_{\mathcal{C}}$ be a \mathbb{J} -filtered type DA-bimodule. Assume that either M is right-bounded or N is left-bounded, so ${}^{\mathcal{A}}M_{\mathcal{B}} \boxtimes_{\mathcal{B}} {}^{\mathcal{B}}N_{\mathcal{C}}$ is defined. Then, ${}^{\mathcal{A}}M_{\mathcal{B}} \boxtimes_{\mathcal{B}} {}^{\mathcal{B}}N_{\mathcal{C}}$ is naturally an $\mathbb{I} \times \mathbb{J}$ -filtered type DA bimodule.*

Proof. This is immediate from the definitions. (See also [LOT14a, Proof of Lemma 2.4].) \square

Remark 2.13. The notion of a filtered A_∞ -module in Definition 2.2 is somewhat restrictive. For example, consider the algebra $\mathcal{A} = \mathbb{F}_2[x]$ and let $M = \mathcal{A}$ as a right \mathcal{A} -module. The two-step filtration $\mathcal{F}_0 M = M \supset \mathcal{F}_1 M = xM$ is not a filtered module in the sense of Definition 2.2.

Remark 2.14. Definitions 2.2, 2.5 and 2.11 look quite similar. Indeed, all are roughly the same as a twisted complex [BK90, Kon95, Sei08] in the relevant dg category. There are some minor differences: unlike [BK90, Definition 1] and [Kon95, p. 15], we allow indexing by arbitrary finite, partially ordered sets, rather than just by \mathbb{Z} ; unlike [Sei08, Section (3l)] we do view the partial ordering as part of the data.

3. CHAIN COMPLEXES OF ATTACHING CIRCLES IN CLOSED SURFACES

Counting pseudo-holomorphic bigons in a symmetric product of a Heegaard surface gives rise to the differential appearing in Heegaard Floer homology. Given a collection of Heegaard tori in the surface, one can generalize these to counts of holomorphic polygons in a natural way. We review these constructions in Subsection 3.1, with a special emphasis on the ‘‘cylindrical reformulation’’ from [Lip06], as that generalizes most readily to the bordered setting (cf. Section 4).

Polygon counts can be organized using the notion of “chain complexes of attaching circles” mentioned in the introduction. In Subsection 3.2 we describe this construction (Definition 3.14), and show how it can be used to construct chain complexes (in the usual sense), as was promised in Proposition 1.3.

In Subsection 3.3, we explain some of the functorial properties of these chain complexes of attaching circles.

In Subsection 3.5 (see especially Proposition 3.52), we show how to glue an \mathbb{I} -filtered chain complex of attaching circles in a Heegaard surface Σ_1 to an \mathbb{J} -filtered chain complex in Σ_2 , to give an $\mathbb{I} \times \mathbb{J}$ -filtered chain complex in the connected sum $\Sigma_1 \# \Sigma_2$. This construction relies on a preliminary construction, *approximation*, introduced in Subsection 3.4 (see especially Definition 3.31). Approximation gives a way of enhancing a chain complex of attaching circles indexed by a set \mathbb{I} to a chain complex of attaching circles indexed by a set $\mathbb{I} \times \mathbb{J}$. Approximations are constructed in Proposition 3.39.

In Subsection 3.6, it is shown that the constructions given here (specifically, the filtered complex associated to a chain complex of attaching circles gotten by gluing) generalize the construction of the filtered complex associated to the branched double cover of a link from [OSz05].

3.1. Holomorphic curves in Heegaard multi-diagrams.

Definition 3.1. *Let Σ be a compact, oriented surface without boundary of some genus g , equipped with a basepoint $z \in \Sigma$. A complete set of attaching circles is a collection $\beta = \{\beta_1, \dots, \beta_g\}$ of homologically independent, pairwise disjoint embedded circles in $\Sigma \setminus z$. A pointed Heegaard multi-diagram is a surface Σ equipped with some number n of complete sets of attaching circles.*

Definition 3.2. *Let \mathbb{I} be a finite, partially ordered set. An \mathbb{I} -filtered admissible collection of attaching circles is a collection $\{\beta^i\}_{i \in \mathbb{I}}$ of g -tuples of attaching circles β^i with the property that for each sequence $i_1 < \dots < i_n$ in \mathbb{I} , the Heegaard multi-diagram $(\Sigma, \beta^{i_1}, \dots, \beta^{i_n}, z)$ is weakly admissible for all spin^c -structures in the sense of [OSz04b, Definition 4.10].*

We will be primarily interested in the case where $\mathbb{I} = \{0, 1\}^n$.

Given two pairs of complete sets of attaching circles $\alpha = \{\alpha_1, \dots, \alpha_g\}$ and $\beta = \{\beta_1, \dots, \beta_g\}$ so that $(\Sigma, \alpha, \beta, z)$ is weakly admissible for all spin^c -structures, let $\widehat{CF}(\alpha, \beta, z)$ denote the Lagrangian intersection Floer complex of the tori $T_\alpha = \alpha_1 \times \dots \times \alpha_g$ and $T_\beta = \beta_1 \times \dots \times \beta_g$ in $\text{Sym}^g(\Sigma \setminus \{z\})$. The complex $\widehat{CF}(\alpha, \beta, z)$ is generated by $\mathfrak{S}(\alpha, \beta) := T_\alpha \cap T_\beta$.

For an \mathbb{I} -filtered admissible collection of attaching circles and any sequence $i_0 < i_1 < \dots < i_n$ in \mathbb{I} , there is a map

$$(3.3) \quad m_n: \widehat{CF}(\beta^{i_{n-1}}, \beta^{i_n}, z) \otimes \dots \otimes \widehat{CF}(\beta^{i_0}, \beta^{i_1}, z) \rightarrow \widehat{CF}(\beta^{i_0}, \beta^{i_n}, z)$$

defined by counting pseudo-holomorphic polygons; see, for instance, [OSz05, Section 4.2]. In the case where $n = 1$, this is the usual differential on $\widehat{CF}(\beta^{i_0}, \beta^{i_1}, z)$. The maps m_n are well-known to satisfy A_∞ relations; see, for instance, [Sei08] or [FOOO09a, FOOO09b].

Convention 3.4. *We have reversed the order of the arguments to m_n from what is standard in the Heegaard Floer literature, so that m_3 agrees with the standard order for function composition in a category. See also Definition 4.19 and Remark 4.22 for further justification of this choice. We will use the order from Equation 3.3 throughout this paper.*

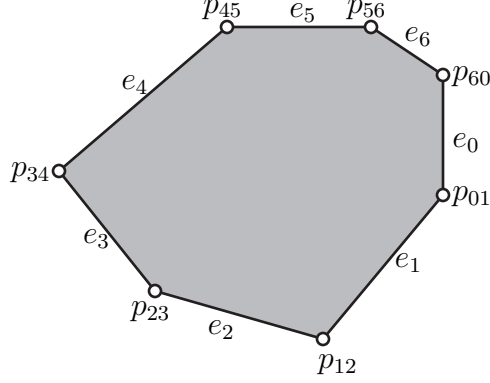


FIGURE 2. **The disk D_7 .** The labeling of the arcs and punctures in the boundary is shown.

Since we are working in the cylindrical formulation of Heegaard Floer homology, we spell out these polygon counts a little more. (The cases of triangles and rectangles were discussed in [Lip06, Section 10].)

Let D_n denote a disk with n labeled punctures on its boundary (a polygon). Label the arcs in ∂D_n as e_0, \dots, e_{n-1} , in clockwise order, and let $p_{i,i+1}$ denote the puncture between e_i and e_{i+1} . (See Figure 2.) Let $\text{Conf}(D_n)$ denote the moduli space of (positively oriented) complex structures on D_n , up to biholomorphism respecting the labeling of the edges. (For $n \geq 3$, $\text{Conf}(D_n)$ is an $(n-3)$ -dimensional ball.) The space $\text{Conf}(D_n)$ has a natural Deligne-Mumford compactification $\overline{\text{Conf}}(D_n)$, which is diffeomorphic to the associahedron (see, e.g., [Dev99]). The boundary $\overline{\text{Conf}}(D_n) \setminus \text{Conf}(D_n)$ of $\overline{\text{Conf}}(D_n)$ consists of trees of polygons.

Points of $\text{Conf}(D_n)$ are equivalence classes $[j]$ of complex structures j on D_n . We would like to view elements of $\text{Conf}(D_n)$ as honest complex structures, instead of equivalence classes, so we explain how to do so. There is an infinite-dimensional bundle $\mathcal{E}_n \rightarrow \text{Conf}(D_n)$ of honest complex structures j . Since $\text{Conf}(D_n)$ is contractible, this bundle is necessarily trivial and, in particular, admits a section. Less trivially, if we replace \mathcal{E}_n with the bundle of complex structures together with choices of strip-like ends (in the sense of [Sei08, Section 9(a)]), which we still denote \mathcal{E}_n , then the projection map $\mathcal{E}_n \rightarrow \text{Conf}(D_n)$ extends to a continuous map $\overline{\mathcal{E}}_n \rightarrow \overline{\text{Conf}}(D_n)$, where $\overline{\mathcal{E}}_n$ is a partial compactification of \mathcal{E}_n given as follows: a point $p \in \partial \overline{\text{Conf}}(D_n)$ corresponds to a tree of disks together with an equivalence class of complex structures on each of those disks, and the fiber of $\overline{\mathcal{E}}_n$ over p is the space of complex structures on those disks inducing the specified equivalence class, together with a choice of strip-like ends for each of the disks. Seidel proves that one can choose sections of $\overline{\mathcal{E}}_n \rightarrow \overline{\text{Conf}}(D_n)$, for each n , which agree at the boundary strata [Sei08, Lemma 9.3]. For the rest of the paper, we will identify $\overline{\text{Conf}}(D_n)$ with its image in $\overline{\mathcal{E}}_n$ under the chosen section, and view elements of $\text{Conf}(D_n)$ as honest complex structures, not equivalence classes of them.

Similarly, fix a family ω_j , $j \in \text{Conf}(D_n)$ of symplectic forms on D_n with cylindrical ends, so that ω_j is adjusted to j [BEH⁺03, Section 3.3], and so that as j approaches the boundary of $\overline{\text{Conf}}(D_n)$, the ω_j split as in (a very simple case of) symplectic field theory [BEH⁺03, Section 3.3] to the corresponding pairs of symplectic forms on $D_m \amalg D_{n+2-m}$.

Definition 3.5. Fix a symplectic form ω_Σ on Σ . An admissible collection of almost-complex structures consists of

- a choice of \mathbb{R} -invariant almost-complex structure J on $\Sigma \times [0, 1] \times \mathbb{R}$ of the kind used to compute Heegaard Floer homology (i.e., satisfying [Lip06, Conditions (J1)–(J5)]), and
- a choice of a smooth family $\{J_j\}_{j \in \text{Conf}(D_n)}$ of almost-complex structures on $\Sigma \times D_n$, for each $n \geq 3$,

satisfying the following conditions:

(J-1) For each $j \in \text{Conf}(D_n)$, the projection map

$$\pi_{\mathbb{D}}: \Sigma \times D_n \rightarrow D_n$$

is (J_j, j) -holomorphic.

(J-2) For each $j \in \text{Conf}(D_n)$, the fibers of $\pi_{\mathbb{D}}$ are J_j -holomorphic.

(J-3) Each almost-complex structure J_j is adjusted to the split symplectic form $\omega_\Sigma \oplus \omega_j$ on $\Sigma \times D_n$.

(J-4) Each almost-complex structure J_j agrees with J near the punctures of D_n , in the sense that each puncture has a strip-like neighborhood U in D_n so that $(\Sigma \times U, J_j|_{\Sigma \times U})$ is bi-holomorphic to $(\Sigma \times [0, 1] \times (0, \infty), J)$.

(J-5) Suppose that $\{j_\alpha\} \subset \text{Conf}(D_n)$ is a sequence converging to a point $j_\infty \in \partial \overline{\text{Conf}}(D_n)$. For notational simplicity, suppose j_∞ lies in the codimension-one boundary, and so corresponds to a point $(j_{\infty,1}, j_{\infty,2}) \in \text{Conf}(D_{m+1}) \times \text{Conf}(D_{n-m+1})$. Then the complex structures J_{j_α} are required to converge to the complex structure $J_{j_{\infty,1}} \amalg J_{j_{\infty,2}}$ on $(\Sigma \times D_{m+1}) \amalg (\Sigma \times D_{n-m+1})$.

(Convergence of the J_{j_α} means the following. As $\alpha \rightarrow \infty$, certain arcs in D_{m+1} collapse. Over neighborhoods of these arcs, the complex structures J_{j_α} should be obtained by inserting longer and longer necks, as in [BEH⁺03, Section 3.4]. Outside these neighborhoods, we require convergence in the C^∞ -topology.)

We require the analogous compatibility condition for the higher-codimension boundary of $\overline{\text{Conf}}(D_n)$, as well.

Definition 3.6. Let $\{J_j\}_{j \in \text{Conf}(D_n)}$ be an admissible collection of almost-complex structures. Given g -tuples of attaching circles β^0, \dots, β^n and generators $\mathbf{x}^{i,i+1} \in \widehat{CF}(\beta^i, \beta^{i+1}, z)$ (with the understanding that $\beta^{n+1} = \beta^0$), consider surfaces S with boundary and boundary punctures, and proper maps

$$(3.7) \quad u: (S, \partial S) \rightarrow (\Sigma \times D_{n+1}, (\beta^0 \times e_0) \cup \dots \cup (\beta^n \times e_n))$$

asymptotic to $\mathbf{x}^{i,i+1}$ at $p_{i,i+1}$. This space decomposes into homology classes (compare [Lip06, Sections 2 and 10.1.1]); let $\pi_2(\mathbf{x}^{n,0}, \dots, \mathbf{x}^{0,1})$ denote the set of homology classes of such maps.

Let $\mathcal{M}(\mathbf{x}^{n,0}, \mathbf{x}^{n-1,n}, \dots, \mathbf{x}^{0,1})$ denote the moduli space of pairs (j, u) where $j \in \text{Conf}(D_{n+1})$ and u is a map as in Formula (3.7) such that:

- (M-0) The image of u is disjoint from $\{z\} \times D_{n+1}$ (i.e., has multiplicity 0 at z).
- (M-1) u is (i, J_j) -holomorphic for some complex structure i on S ,
- (M-2) $\pi_{\mathbb{D}} \circ u$ is a g -fold branched cover, and
- (M-3) u is an embedding.

We will often abuse notation and write elements of $\mathcal{M}(\mathbf{x}^{n,0}, \mathbf{x}^{n-1,n}, \dots, \mathbf{x}^{0,1})$ as maps u , but the complex structure j is part of the data.

The space $\mathcal{M}(\mathbf{x}^{n,0}, \dots, \mathbf{x}^{0,1})$ decomposes as a disjoint union

$$\mathcal{M}(\mathbf{x}^{n,0}, \mathbf{x}^{n-1,n}, \dots, \mathbf{x}^{0,1}) = \coprod_{B \in \pi_2(\mathbf{x}^{n,0}, \dots, \mathbf{x}^{0,1})} \mathcal{M}^B(\mathbf{x}^{n,0}, \mathbf{x}^{n-1,n}, \dots, \mathbf{x}^{0,1}).$$

We will often abbreviate $\mathcal{M}^B(\mathbf{x}^{n,0}, \mathbf{x}^{n-1,n}, \dots, \mathbf{x}^{0,1})$ to \mathcal{M}^B .

For each $B \in \pi_2(\mathbf{x}^{n,0}, \dots, \mathbf{x}^{0,1})$ the space $\mathcal{M}^B(\mathbf{x}^{n,0}, \dots, \mathbf{x}^{0,1})$ has a well-defined expected dimension $\text{ind}(B) + n - 2$.

Proposition 3.8. *Admissible collections of almost-complex structures exist. Moreover, with respect to a generic admissible collection of almost-complex structures, each of the moduli spaces $\mathcal{M}^B(\mathbf{x}^{n,0}, \dots, \mathbf{x}^{0,1})$ is transversely cut out by the $\bar{\partial}$ -operator, and hence is a smooth manifold of dimension $\text{ind}(B) + n - 2$.*

Proof. The first part of the statement (existence) follows from the observation that the space of almost-complex structures satisfying Conditions (J-1)–(J-4) is contractible, so the extension problem specified by Condition (J-5) has a solution. The second part (transversality) follows by a similar argument to [Lip06, Section 3]. \square

Remark 3.9. In general, of course, the moduli spaces $\mathcal{M}^B(\mathbf{x}^{n,0}, \dots, \mathbf{x}^{0,1})$ are non-compact, though they admit compactifications $\overline{\mathcal{M}}^B(\mathbf{x}^{n,0}, \dots, \mathbf{x}^{0,1})$ in terms of trees of holomorphic curves. For generic admissible collections of almost-complex structures, if $\text{ind}(B) = -n + 2$ then $\mathcal{M}^B(\mathbf{x}^{n,0}, \dots, \mathbf{x}^{0,1})$ is a compact 0-manifold: all broken curves in $\overline{\mathcal{M}}^B(\mathbf{x}^{n,0}, \dots, \mathbf{x}^{0,1})$ belong to negative expected dimension families, and hence by transversality do not occur.

Definition 3.10. *Let $(\Sigma, \beta^0, \dots, \beta^n, z)$ be a weakly admissible Heegaard multi-diagram. Define a map*

$$m_n: \widehat{CF}(\beta^{n-1}, \beta^n, z) \otimes \widehat{CF}(\beta^{n-2}, \beta^{n-1}, z) \otimes \dots \otimes \widehat{CF}(\beta^0, \beta^1, z) \rightarrow \widehat{CF}(\beta^0, \beta^n, z)$$

as follows. Choose a generic admissible collection of almost-complex structures (as guaranteed by Proposition 3.8), and define

$$m_n(\mathbf{x}^{n-1,n}, \dots, \mathbf{x}^{0,1}) = \sum_{\mathbf{x}^{0,n} \in \mathfrak{S}(\beta^0, \beta^n)} \sum_{\substack{B \in \pi_2(\mathbf{x}^{n,0}, \mathbf{x}^{n-1,n}, \dots, \mathbf{x}^{0,1}) \\ \text{ind}(B) = 3-n}} (\#\mathcal{M}^B(\mathbf{x}^{n,0}, \mathbf{x}^{n-1,n}, \dots, \mathbf{x}^{0,1})) \mathbf{x}^{0,n}.$$

(Here, $\mathbf{x}^{n,0}$ is the generator of $\widehat{CF}(\beta^n, \beta^0, z)$ corresponding to $\mathbf{x}^{0,n} \in \widehat{CF}(\beta^0, \beta^n, z)$.) As explained Remark 3.9, each moduli space being counted is finite, and it follows from weak admissibility of the Heegaard multi-diagram that the sum itself is finite.

Note that when $n = 1$ the map m_n is just the differential on $\widehat{CF}(\beta^0, \beta^1, z)$.

Proposition 3.11. *Let β^0, \dots, β^n be g -tuples of attaching circles in Σ . Then the maps m_n satisfy the following A_∞ relation:*

$$(3.12) \quad \sum_{i=1}^n \sum_{j=1}^{n-i-1} m_{n-j+1}(\mathbf{x}^{n-1,n}, \dots, \mathbf{x}^{i+j,i+j+1}, m_j(\mathbf{x}^{i-1,i}, \dots, \mathbf{x}^{i+j-1,i+j}), \mathbf{x}^{i-2,i-1}, \dots, \mathbf{x}^{0,1}) = 0.$$

Proof. This follows in the usual way, by considering the ends of the 1-dimensional moduli space

$$\bigcup_{\substack{B \in \pi_2(\mathbf{x}^{n,0}, \dots, \mathbf{x}^{0,1}) \\ \text{ind}(B) = 4-n}} \#\mathcal{M}^B(\mathbf{x}^{n,0}, \mathbf{x}^{n-1,n}, \dots, \mathbf{x}^{0,1}).$$

Conditions (J-1), (J-2), (J-3) and (J-4) guarantee that this moduli space has a compactification in terms of broken holomorphic polygons. (See [BEH⁺03] or [Abb14]. In particular, by Condition (J-5), approaching the boundary of $\text{Conf}(D_n)$ has the effect of splitting along a hypersurface, as in [BEH⁺03, Theorem 10.3].) Condition (J-5) allows us to identify the counts of these broken polygons with the counts used to define m_{n-j+1} and m_j . \square

We conclude this section by noting that, via the *tautological correspondence*, the maps m_n can also be defined by counting polygons in the symmetric product. That is, given a family J_j , $j \in \text{Conf}(D_n)$, of admissible almost-complex structures there is a corresponding family $\text{Sym}^g(J_j)$, $j \in \text{Conf}(D_n)$ of maps from D_n to the space of almost-complex structures on the symmetric product $\text{Sym}^g(\Sigma)$. Given a J_j -holomorphic map u as in Formula (3.7) there is a corresponding $\text{Sym}^g(J_j)$ -holomorphic map $\phi_u: D_n \rightarrow \text{Sym}^g(\Sigma)$ defined by $\phi_u(x) = (\pi_\Sigma \circ u)((\pi_{\mathbb{D}} \circ u)^{-1}(x))$. The map ϕ_u sends the edges of D_n to $T_{\beta^1}, T_{\beta^2}, \dots, T_{\beta^n}$, where T_{β^i} is the image of $\beta_1^i \times \dots \times \beta_g^i \subset \Sigma^{\times g}$ in $\text{Sym}^g(\Sigma)$.

Lemma 3.13. *The assignment $u \mapsto \phi_u$ gives a bijection between the moduli space of polygons as in Definition 3.6 and the moduli space of polygons in $(\text{Sym}^g(\Sigma), T_{\beta^1}, \dots, T_{\beta^n})$ which are holomorphic with respect to one of the complex structures $\text{Sym}^g(J_j)$, $j \in \text{Conf}(D_n)$. In particular, the maps m_n from Definition 3.11 agree with the maps defined by counting holomorphic polygons in the symmetric product as in, e.g., [OSz04b, Section 8] or [OSz05, Section 4.2].*

Proof. The proof is the same as the proof for bigons [Lip06, Proposition 13.6], with only notational changes. \square

3.2. Chain complexes of attaching circles: definition.

Definition 3.14. *Let \mathbb{I} be a finite, partially ordered set and let (Σ, z) be a pointed closed, oriented surface. An \mathbb{I} -filtered chain complex of attaching circles (or simply chain complex of attaching circles) consists of the following data:*

- an admissible collection of attaching circles $\{\beta^i\}_{i \in \mathbb{I}}$ and
- for each pair of elements $i_1 < i_2$ in \mathbb{I} , a chain $\eta^{i_1 < i_2} \in \widehat{CF}(\beta^{i_1}, \beta^{i_2}, z)$.

The chains $\eta^{i < j}$ are required to satisfy the following compatibility conditions, indexed by pairs $i, j \in \mathbb{I}$ with $i < j$:

$$(3.15) \quad \sum_{n=1}^{\infty} \sum_{i=i_0 < i_1 < \dots < i_{n-1} < i_n=j} m_n(\eta^{i_{n-1} < i_n}, \dots, \eta^{i_0 < i_1}) = 0,$$

where the sum is taken over the sequences i_1, \dots, i_{n-1} and m_n denotes the counts of holomorphic $n+1$ -gons (Definition 3.10). We will usually suppress \mathbb{I} and Σ from the notation and write a chain complex of attaching circles as a triple $(\{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i_1 < i_2}\}_{i_1, i_2 \in \mathbb{I}}, z)$.

For example, Equation (3.15) implies that if $i < j$ are consecutive (i.e. there is no other $k \in \mathbb{I}$ between i and j), then $\eta^{i < j}$ is a cycle.

Remark 3.16. The attentive reader might notice a similarity between Definition 3.14 and Definition 2.1 (say). Indeed, let TFuk denote the full subcategory of the Fukaya category of $\text{Sym}^g(\Sigma \setminus \{z\})$ generated by Heegaard tori. Then a chain complex of attaching circles is just a twisted complex (type D structure) in TFuk (modulo the caveats in Remarks 2.6 and 2.14).

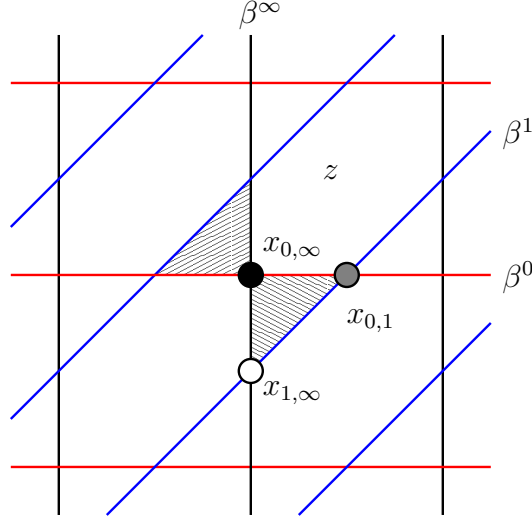


FIGURE 3. **Example of a chain complex of attaching circles.** The picture takes place in the torus, with three curves β^0 , β^1 , and β^∞ , which pairwise intersect in three points $x_{0,\infty}$, $x_{0,1}$, and $x_{1,\infty}$. The fact that $m_2(x_{1,\infty}, x_{0,1}) = 0$ is illustrated by the two hatched (canceling) triangles. Thus, in this model, the chains $\eta^{0,1} = x_{0,1}$, $\eta^{1,\infty} = x_{1,\infty}$ and $\eta^{0,\infty} = 0$ give a chain complex of attaching circles.

The following example plays a pivotal role in the proof of the surgery exact triangle for Heegaard Floer homology [OSz04a, Theorem 9.1].

Example 3.17. Let $\mathbb{I} = \{0, 1, \infty\}$ with the obvious ordering, and let β^0 , β^1 , and β^∞ be three sets of attaching circles in a genus one surface, with the property that $(\Sigma, \beta^0, \beta^1, \beta^\infty, z)$ is a Heegaard triple representing $\overline{\mathbb{C}P}^2$ (i.e., with β^0 , β^1 , β^∞ at slopes 0, 1, ∞ , respectively). Choose $\eta^{0,1}$ and $\eta^{1,\infty}$ to be cycles representing the non-trivial homology class in

$$\widehat{HF}(\beta^0, \beta^1, z) = \widehat{HF}(S^3) \cong \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad \widehat{HF}(\beta^1, \beta^\infty, z) = \widehat{HF}(S^3) \cong \mathbb{Z}/2\mathbb{Z},$$

respectively. We can find a chain $\eta^{0,\infty}$ so that $d\eta^{0,\infty} = m_2(\eta^{1,\infty}, \eta^{0,1})$. The data $(\{\beta^0, \beta^1, \beta^\infty\}, \{\eta^{0,1}, \eta^{1,\infty}, \eta^{0,\infty}\})$ forms a chain complex of attaching circles. (See Figure 3.)

Chain complexes of attaching circles can be used to turn sets of attaching circles into chain complexes, using the following Yoneda embedding.

Definition 3.18. Suppose that $(\{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i_1 < i_2}\}_{i_1, i_2 \in \mathbb{I}}, z)$ is a chain complex of attaching circles, and α is an additional set of attaching circles with the property that for all sequences $i_1 < \dots < i_n$ in \mathbb{I} , the multi-diagram $(\Sigma, \alpha, \beta^{i_1}, \dots, \beta^{i_n}, z)$ is weakly admissible. We call the collection of chain complexes

$$\{\widehat{CF}(\alpha, \beta^i, z)\}_{i \in \mathbb{I}}$$

equipped with the morphisms

$$D^{i < j}(\mathbf{x}) = \sum_{i=i_1 < \dots < i_n=j} m_n(\eta^{i_{n-1} < i_n}, \dots, \eta^{i_1 < i_2}, \mathbf{x})$$

where the sum is over all subsequences of \mathbb{I} starting at i and ending at j the Heegaard Floer complex associated to α and $(\{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i_1 < i_2}\}_{i_1, i_2 \in \mathbb{I}})$. We denote this filtered chain complex by $\widehat{CF}(\alpha, \{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i_1 < i_2}\}_{i_1, i_2 \in \mathbb{I}}, z)$.

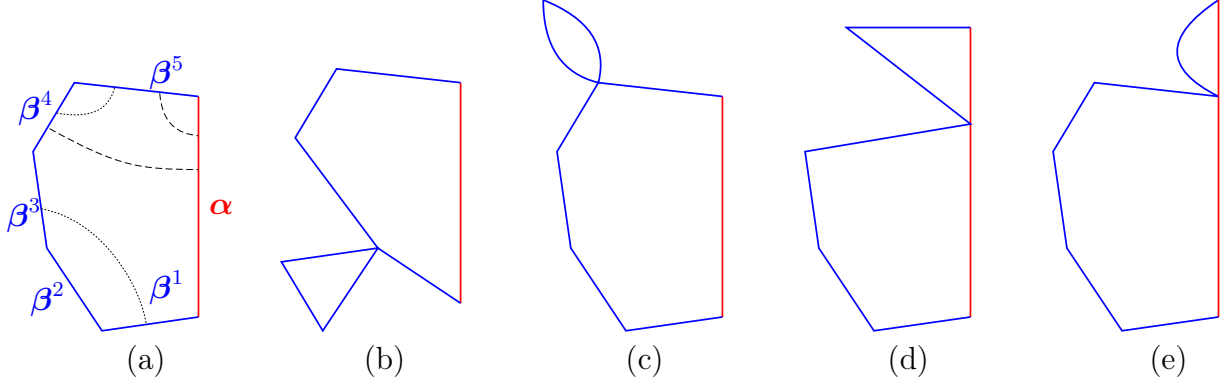


FIGURE 4. **Proof of Proposition 3.19.** The hexagon illustrated in (a) can be degenerated along one of the dotted lines to give (b) or (c). Degenerating along the dashed lines gives one of (d) or (e). Degenerations of type (b) and (c) cancel in Formula (3.15); those of type (d) give terms of the form $D^{j<k} \circ D^{i<j}$; and those of type (e) give terms in $dD^{i<k}$.

The terminology is justified by the following more precise version of Proposition 1.3:

Proposition 3.19. *Let α be a set of attaching circles and $(\{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i_1 < i_2}\}_{i_1, i_2 \in \mathbb{I}}, z)$ be a chain complex of attaching circles. Suppose that for all sequences $i_1 < \dots < i_n$ in \mathbb{I} , the multi-diagram $(\Sigma, \alpha, \beta^{i_1}, \dots, \beta^{i_n}, z)$ is weakly admissible. Then the Heegaard Floer complex associated to α and $(\{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i_1 < i_2}\}_{i_1, i_2 \in \mathbb{I}}, z)$ is an \mathbb{I} -filtered chain complex in the sense of Definition 2.1.*

Proof. As we will see, the structure equation is an easy consequence of the associativity formula for counts of holomorphic polygons, Proposition 3.11. (Compare Figure 4.) For any $i < k$ we have

$$(3.20) \quad \sum_{i < j < k} D^{j < k} (D^{i < j}(\mathbf{x})) = \sum_{i=i_0 < \dots < i_m = j < \dots < i_n = k} m(\eta^{i_{n-1} < i_n}, \dots, \eta^{i_{m+1} < i_m}, m(\eta^{i_{m-1} < i_m}, \dots, \eta^{i_0 < i_1}, \mathbf{x}))$$

(omitting the indices on the m 's), while

$$(3.21) \quad d(D^{i < k})(\mathbf{x}) = \sum_{i=i_0 < \dots < i_n = k} m_1(m_n(\eta^{i_{n-1} < i_n}, \dots, \eta^{i_0 < i_1}, \mathbf{x})) + m_n(\eta^{i_{n-1} < i_n}, \dots, \eta^{i_0 < i_1}, m_1(\mathbf{x})).$$

Equation (3.15) gives

$$(3.22) \quad 0 = \sum_{i=i_0 < \dots < i_n = k} \sum_{1 \leq j \leq \ell < n} m(\eta^{i_{n-1} < i_n}, \dots, m(\eta^{i_{\ell-1} < i_\ell}, \dots, \eta^{i_j < i_{j+1}}), \dots, \eta^{i_0 < i_1}, \mathbf{x}).$$

The sum of the right hand sides of Equations (3.20), (3.21) and (3.22) is the left hand side of Equation (3.12), and hence is equal to zero. Thus,

$$d(D^{i < k}) + \sum_{\{j | i < j < k\}} D^{j < k} \circ D^{i < j} = 0. \quad \square$$

3.3. Morphisms between chain complexes of attaching circles. The next step in developing the theory of chain complexes of attaching circles is to verify that the chain complex considered in Proposition 3.19 is invariant under change of admissible collection of almost-complex structures and isotopies of the α - and β -circles. The argument is based on the usual “continuation maps” in Floer homology, and is similar to the proofs in [Rob08] and [Bal11]; see also [Sei08, Section 10(a)] in the more general setting of Fukaya categories. In this section we will prove invariance under isotopies, leaving invariance under changes of admissible collection of almost-complex structures as an exercise to the reader.

So, fix an \mathbb{I} -filtered admissible collection of attaching circles $\{\beta^i\}_{i \in \mathbb{I}}$ in a pointed surface (Σ, z) , an element $k \in \mathbb{I}$, and a collection of attaching circles γ^k Hamiltonian isotopic to β^k . Let $\gamma^i = \beta^i$ for $i \neq k$. We will assume that the γ^k are close enough to the β^k in a sense that will be made precise in two places below; in practice, by breaking a Hamiltonian isotopy up into a sequence of smaller isotopies, this closeness assumption can always be achieved. Fix also an admissible collection of almost-complex structures, chosen generically in the sense of Proposition 3.8, and so that the moduli spaces described below are transversally cut out. The argument that such a family of almost-complex structures exists is similar to the (largely omitted) proof of Proposition 3.8.

Our first goal is to define maps

$$f_n: \widehat{CF}(\beta^{i_{n-1}}, \beta^{i_n}, z) \otimes \cdots \otimes \widehat{CF}(\beta^{i_0}, \beta^{i_1}, z) \rightarrow \widehat{CF}(\gamma^{i_0}, \gamma^{i_n}, z)$$

satisfying the A_∞ -homomorphism relation, i.e., so that

$$(3.23) \quad 0 = \sum_{1 \leq a \leq b \leq m} f_{m-b+a}(x_m, x_{m-1}, \dots, m_{b-a+1}(x_b, \dots, x_a), \dots, x_1) \\ + \sum_c \sum_{n_1 + \dots + n_c = m} m_c(f_{n_1}(x_m, x_{m-1}, \dots, x_{m-n_1+1}), \\ f_{n_2}(x_{m-n_1}, \dots, x_{m-n_1-n_2+1}), \dots, f_{n_c}(x_{n_c}, \dots, x_1))$$

for any sequence of sets of attaching circles $\beta^{i_0}, \dots, \beta^{i_m}$ and elements $x_j \in \widehat{CF}(\beta^{i_{j-1}}, \beta^{i_j}, z)$.

The maps f_n are defined as follows:

- (f-1) If $n = 1$ and $k \notin \{i_0, i_1\}$ then $f_1: \widehat{CF}(\beta^{i_0}, \beta^{i_1}, z) \rightarrow \widehat{CF}(\gamma^{i_0}, \gamma^{i_1}, z) = \widehat{CF}(\beta^{i_0}, \beta^{i_1}, z)$ is the identity map.
- (f-2) If $n > 1$ and $k \notin \{i_0, \dots, i_n\}$ then $f_n = 0$.
- (f-3) For any $n \geq 1$, if $k = i_0$ then f_n is defined by counting holomorphic $(n+1)$ -gons with dynamic boundary conditions along one edge. That is, we count holomorphic maps

$$u: (S, \partial S) \rightarrow ((\Sigma \setminus \{z\}) \times P, L_C \cup (\beta^{i_1} \times e_2) \cup \cdots \cup \beta^{i_n} \times e_{n+1})$$

where $C \in \mathbb{R}$ is allowed to vary and L_C is given as follows. For each $j \in \text{Conf}(D_n)$, fix an identification of a neighborhood of the edge e_1 in P with $\mathbb{R} \times [0, \epsilon)$ so that the symplectic form ω_j on P is the pullback of the usual symplectic form on \mathbb{R}^2 . We require these identifications to be continuous, consistent with the cylindrical ends, and consistent across strata of $\text{Conf}(n+1)$; compare [Sei08, Sections (9e)–(9i) and (10e)]. For each j , fix a Hamiltonian isotopy ϕ_t , $t \in [0, 1]$, from β^k to γ^k , induced by

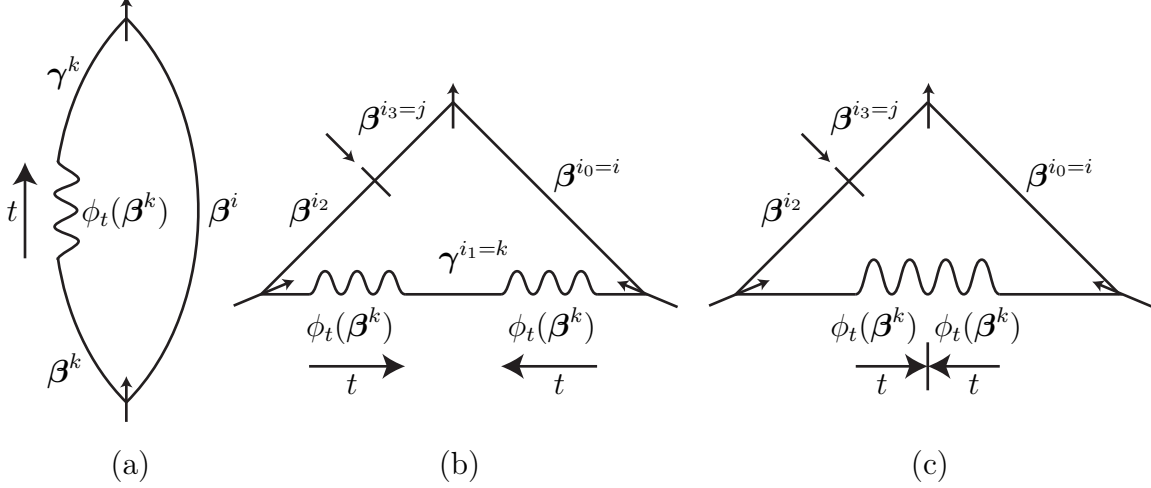


FIGURE 5. **Continuation maps associated to a Hamiltonian isotopy.** (a) A polygon defining Case (f-3) of the map f_1 . (b) and (c) Examples of polygons defining Case (f-5) of the map f_3 for $k = i_1$. (b) represents to the boundary conditions in Formula (3.26) while (c) represents the boundary conditions in Formula (3.27).

a time-dependent Hamiltonian H_t . Then

$$(3.24) \quad L_C = [\beta^k \times (-\infty, C] \times \{0\}] \cup \{(\phi_t(\beta^k), C + t, 0) \mid t \in [0, 1]\} \\ \cup [\gamma^k \times [C + 1, \infty) \times \{0\}].$$

The manifold L_C is not Lagrangian with respect to $\omega = \omega_\Sigma \times \omega_j$, but rather with respect to the deformed form

$$(3.25) \quad \omega - (d(\psi(s)H_t)) \wedge dt,$$

where $\psi: [0, \epsilon] \rightarrow \mathbb{R}$ is a smooth cut-off function taking the value 1 on a neighborhood of 0 and 0 on a neighborhood of ϵ . Assuming that the Hamiltonian isotopy H_t is small enough, this deformed form is still symplectic and still tames the almost-complex structures J_j under consideration. This is one of the two “close” requirements we place on the β^i and γ^i .

- (f-4) For any $n \geq 1$, if $k = i_n$ then f_n is defined similarly to the previous case, but with dynamic boundary conditions along the edge e_{n+1} .
- (f-5) For any $n \geq 2$, if $k = i_j$ for some $0 < j < n$ then f_n is defined by counting holomorphic $(n + 1)$ -gons

$$u: (S, \partial S) \rightarrow ((\Sigma \setminus \{z\}) \times P, (\beta^{i_0} \times e_1) \cup \dots \cup L'_s \cup \dots \cup \beta^{i_n} \times e_{n+1})$$

for some $s \in (-\infty, 1]$ where the boundary condition L'_s along the edge e_{j+1} is given as follows. For each $j \in \text{Conf}(D_n)$, fix an identification of a neighborhood of the edge e_{j+1} with $\mathbb{R} \times [0, \epsilon)$, as in Case (f-3). Then for $s < 0$,

$$(3.26) \quad L'_s = [\beta^k \times (-\infty, s - 1] \times \{0\}] \cup \{(\phi_t(\beta^k), s + t - 1, 0) \mid t \in [0, 1]\} \\ \cup [\gamma^k \times [s, -s] \times \{0\}] \cup \{(\phi_{1-t}(\beta^k), -s + t, 0) \mid t \in [0, 1]\} \\ \cup [\beta^k \times [-s + 1, \infty) \times \{0\}].$$

For $s \in [0, 1]$,

$$(3.27) \quad L'_s = [\beta^k \times (-\infty, s-1] \times \{0\}] \cup \{(\phi_t(\beta^k), s+t-1, 0) \mid t \in [0, 1-s]\} \\ \cup \{(\phi_{1-t}(\beta^k), -s+t, 0) \mid t \in [s, 1]\} \cup [\beta^k \times [1-s, \infty) \times \{0\}].$$

See Figure 5. (Again, the submanifolds L'_s are not Lagrangian with respect to ω , but rather with respect to a deformed symplectic form as in Equation (3.25). This is the second of the two “close” requirement we place on β^i and γ^i .)

Lemma 3.28. *The maps f_n satisfy the A_∞ -homomorphism relation (3.23).*

Proof. There are several cases. The case that $k \notin \{i_0, \dots, i_m\}$ is trivial. Next, consider the case that $k = i_0$. One-dimensional moduli spaces of polygons as in Case (f-3) of the definition of f_m have three kinds of ends:

- Breakings of polygons where the edge e_1 does not break. These correspond to the terms in the first sum in Equation (3.23) with $i > 1$.
- Breakings of polygons where the edge e_1 breaks below $[C, C+1]$. These correspond to the terms in the first sum in Equation (3.23) with $i = 1$.
- Breakings of polygons where the edge e_1 breaks above $[C, C+1]$. These correspond to the terms in the second sum in Equation (3.23).

The case that $k = i_m$ is similar to the case that $k = i_0$.

Finally, consider the case that $k = i_j$ for some $0 < j < m$. One-dimensional moduli spaces of polygons as in Case (f-5) of the definition of f_m have five kinds of ends, as shown in Figure 6. The ends correspond to terms in Equation (3.23), as follows:

- (1) Ends of type (b) correspond to terms in the first sum where neither x_k nor x_{k+1} is an input to the multiplication m .
- (2) Ends of type (c) correspond to terms in the second sum where x_k and x_{k+1} are inputs to the same map f .
- (3) Ends of type (d) correspond to terms in the first sum where exactly one of x_k and x_{k+1} is an input to the multiplication m .
- (4) Ends of type (e) (which correspond to $s = 1$) correspond to terms in the first sum where x_k and x_{k+1} are both input to the multiplication m . (Note that in this case, we must have $m - j + i = 1$.)
- (5) Ends of type (f) (which correspond to $s = -\infty$) correspond to terms in the second sum where x_k and x_{k+1} are inputs to different f 's. (Note that the symmetry in the definition of L'_s forces degenerations to occur at two corners at once.) \square

Next, we turn to the continuation maps for chain complexes of attaching circles. With notation as above, suppose that we are given chains $\eta^{i_1 < i_2}$ making $(\{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i < j}\}_{i, j \in \mathbb{I}}, z)$ into a chain complex of attaching circles. Let

$$\zeta^{i < j} = \sum_{i=i_0 < i_1 < \dots < i_n=j} f_n(\eta^{i_{n-1} < i_n}, \eta^{i_{n-2} < i_{n-1}} \dots, \eta^{i_0 < i_1}).$$

Lemma 3.29. *The data $(\{\gamma^i\}_{i \in \mathbb{I}}, \{\zeta^{i < j}\}_{i < j \in \mathbb{I}}, z)$ forms a chain complex of attaching circles.*

Proof. This is straightforward from the definitions and Lemma 3.28. \square

Proposition 3.30. *Let $(\{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i_1 < i_2}\}_{i_1, i_2 \in \mathbb{I}}, z)$ be a chain complex of attaching circles, ϕ_t an exact Hamiltonian isotopy, and $\{\gamma^i\}_{i \in \mathbb{I}}$ the new collection of attaching circles gotten*

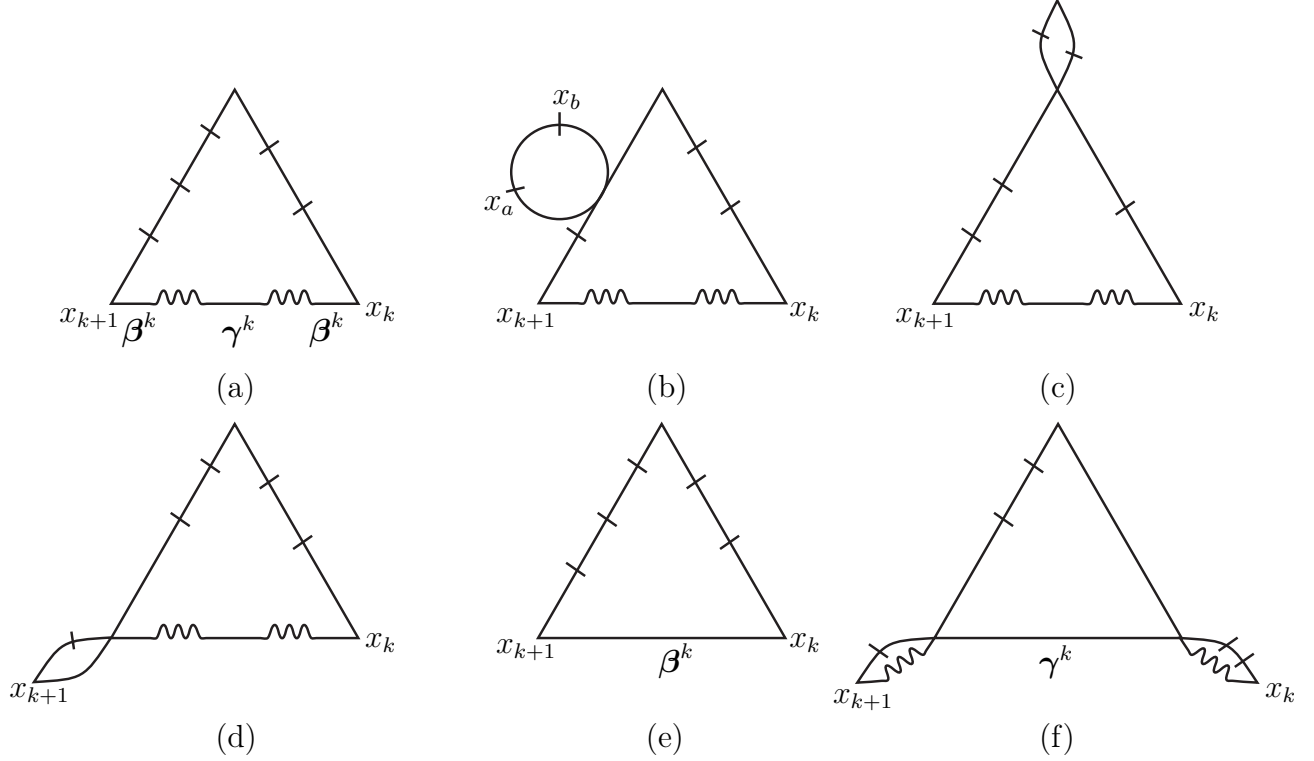


FIGURE 6. **Proof that f satisfies the A_∞ relation.** (a) shows an 8-gon: the output corner and the corners corresponding to x_k and x_{k+1} are drawn as corners, and the rest as tick marks. (b)–(f) show the ways this moduli space of 8-gons can break. For cases (b) and (d) we have only shown one of two cases: the other case is given by reflecting the picture horizontally.

by letting ϕ_1 act on the k^{th} tuple of attaching circles. Let $(\{\gamma^i\}_{i \in \mathbb{I}}, \{\zeta^{i_1 < i_2}\}_{i_1, i_2 \in \mathbb{I}}, z)$ denote the new chain complex of attaching circles as in Lemma 3.29. Then given another g -tuple of attaching circles α , there is a filtered quasi-isomorphism between the associated filtered complexes $\widehat{\mathbf{CF}}(\alpha, \{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i_1 < i_2}\}_{i_1, i_2 \in \mathbb{I}}, z)$ and $\widehat{\mathbf{CF}}(\alpha, \{\gamma^i\}_{i \in \mathbb{I}}, \{\zeta^{i_1 < i_2}\}_{i_1, i_2 \in \mathbb{I}}, z)$ (as defined in Definition 3.18).

Proof. The quasi-isomorphism

$$F: \widehat{\mathbf{CF}}(\alpha, \{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i < j}\}_{i, j \in \mathbb{I}}, z) \rightarrow \widehat{\mathbf{CF}}(\alpha, \{\gamma^i\}_{i \in \mathbb{I}}, \{\zeta^{i < j}\}_{i, j \in \mathbb{I}}, z)$$

is defined similarly to the map f above. Specifically, for $i \leq j \in \mathbb{I}$, define

$$F^{i \leq j}: \widehat{\mathbf{CF}}(\alpha, \beta^i, z) \rightarrow \widehat{\mathbf{CF}}(\alpha, \gamma^j, z)$$

by the following cases:

- (1) If $i = j \neq k$ then $F^{i \leq j}$ is the identity map. (Compare (f-1).)
- (2) If $i < j < k$ or $k < i < j$ then define $F^{i \leq j} = 0$. (Compare (f-2).)
- (3) If $k = i$ then define $F^{i \leq j}$ by counting holomorphic polygons with boundary

$$(\alpha \times e_1) \cup L_C \cup (\gamma^{i_1} \times e_3) \cup \cdots \cup (\gamma^{i_n} \times e_{n+2}),$$

where $i < i_1 < \dots < i_n = j$ is a sequence in \mathbb{I} , asymptotic to $\eta^{i_m < i_{m+1}}$ (or equivalently $\zeta^{i_m < i_{m+1}}$) at the corner between γ^{i_m} and $\gamma^{i_{m+1}}$. Here, L_C is as in Formula (3.24). (Compare (f-3).)

- (4) If $k = j$ then define $F^{i \leq j}$ similarly to the previous case, but with β in place of γ : count polygons with boundary

$$(\alpha \times e_1) \cup (\beta^{i_0} \times e_2) \cup \dots \cup (\beta^{i_{n-1}} \times e_{n+1}) \cup L_C,$$

asymptotic to $\eta^{i_m < i_{m+1}}$ (or equivalently $\zeta^{i_m < i_{m+1}}$) at the corner between β^{i_m} and $\beta^{i_{m+1}}$. (Compare (f-4).)

Note that both this and the previous item cover the case $i = j = k$, and define the same map in this case: it is the usual Floer continuation map associated to the isotopy from β^k to γ^k .

- (5) If $i < k < j$ then define $F^{i \leq j}$ by counting holomorphic polygons with boundary

$$(\alpha \times e_1) \cup (\beta^{i_0} \times e_2) \cup \dots \cup L'_s \cup \dots \cup (\beta^{i_n} \times e_{n+2}),$$

where $i = i_0 < i_1 < \dots < i_n = j$, asymptotic to $\eta^{i_m < i_{m+1}}$ at the corner between β^{i_m} and $\beta^{i_{m+1}}$. Here, L'_s is as defined in Formulas (3.26) and (3.27). (Compare (f-5).)

It is straightforward to verify that F is a chain map. The map of associated graded complexes is the usual Floer continuation map, and hence is a quasi-isomorphism. It follows that F is a quasi-isomorphism, as well. \square

3.4. Close approximations of attaching circles. We would like to describe the gluing of chain complexes of attaching circles which appears in Proposition 1.4. Before doing this, we discuss a preliminary construction which goes into the definition: approximations to attaching circles.

Definition 3.31. Let $\{\beta^i\}_{i \in \mathbb{I}}$ be an admissible collection of attaching circles, and \mathbb{J} another partially ordered set. We say that an $\mathbb{I} \times \mathbb{J}$ -filtered admissible collection of attaching circles $\{\beta^{i \times j}\}_{i \times j \in \mathbb{I} \times \mathbb{J}}$ is an approximation to $\{\beta^i\}_{i \in \mathbb{I}}$ if for each i, j , $\beta^{i \times j}$ is Hamiltonian isotopic to β^i where the Hamiltonian is supported in a tubular neighborhood of β^i .

An approximation is called *efficient* if for each $j_0 < j_1$ in \mathbb{J} and each $i \in \mathbb{I}$, the differential on $\widehat{CF}(\beta^{i \times j_0}, \beta^{i \times j_1}, z)$ vanishes. In particular, it has a unique generator of top degree

$$\Theta^{i \times j_0 < i \times j_1} \in \widehat{CF}(\beta^{i \times j_0}, \beta^{i \times j_1}, z).$$

It is easy to construct efficient approximations; see, for instance, Figure 7.

Let $\{\beta^i\}_{i \in \mathbb{I}}$ be an admissible set of attaching circles and \mathbb{J} another partially ordered set. Let $\{\beta_\epsilon^{i \times j}\}_{i \times j \in \mathbb{I} \times \mathbb{J}}$ be a one-parameter family of efficient approximations, where $\lim_{\epsilon \rightarrow 0} \beta_\epsilon^{i \times j} = \beta^i$ (in the C^∞ topology). Fix a sequence $i_0 \times j_0 < \dots < i_n \times j_n$. For $a_1 \leq \dots \leq a_n$ a non-decreasing sequence, let $R(a_1, \dots, a_n)$ denote the number of repeated entries, counted with multiplicity (so that there are $n + 1 - R(a_1, \dots, a_n)$ distinct entries). Let $k = R(i_0, \dots, i_n)$, and let $i_{s_0} < \dots < i_{s_{n-k}}$ be the subsequence with all but the last of each repeated entry removed.

Convention 3.32. To shorten notation, we will often write ij_k for $i_k \times j_k$.

Suppose that ϵ is sufficiently small. Then given a sequence of $n + 1 - k$ generators $\mathbf{x}_0^{i_{s_0} < i_{s_1}}, \dots, \mathbf{x}_0^{i_{s_{n-k-1}} < i_{s_{n-k}}}$ and $\mathbf{x}_0^{i_{s_0} < i_{s_{n-k}}}$ with the understanding that $\mathbf{x}_0^{i < i'} \in \mathfrak{S}(\beta^i, \beta^{i'})$, there

is a canonically associated sequence of $n + 1$ generators

$$\mathbf{x}_\epsilon^{\dot{ij}_0 < \dot{ij}_1}, \dots, \mathbf{x}_\epsilon^{\dot{ij}_{n-1} < \dot{ij}_n} \text{ and } \mathbf{x}_\epsilon^{\dot{ij}_0 < \dot{ij}_n},$$

with $\mathbf{x}_\epsilon^{\dot{ij}_m < \dot{ij}_{m+1}} \in \mathfrak{S}(\beta_\epsilon^{i \times j}, \beta_\epsilon^{i' \times j'})$ defined by

$$\mathbf{x}_\epsilon^{\dot{ij}_m < \dot{ij}_{m+1}} = \begin{cases} \text{nearest-point to } \mathbf{x}_0^{i_{s_\ell} < i_{s_{\ell+1}}} & \text{if } m = s_\ell \text{ for some } \ell \\ \Theta^{\dot{ij}_m < \dot{ij}_{m+1}} & \text{otherwise (i.e., if } i_m = i_{m+1}) \end{cases}$$

$$\mathbf{x}_\epsilon^{\dot{ij}_0 < \dot{ij}_n} = \text{nearest-point to } \mathbf{x}_0^{i_0 < i_n}.$$

Definition 3.33. Two elements B_1, B_2 of $\pi_2(\mathbf{x}_\epsilon^{\dot{ij}_0 < \dot{ij}_n}, \mathbf{x}_\epsilon^{\dot{ij}_{n-1} < \dot{ij}_n}, \dots, \mathbf{x}_\epsilon^{\dot{ij}_0 < \dot{ij}_1})$ are said to be nearly equivalent if there is a collection of doubly-periodic domains for the Heegaard diagrams $(\beta^{\dot{ij}_m}, \beta^{\dot{ij}_{m+1}})$ for various m with $i_m = i_{m+1}$ which can be added to B_2 to get B_1 . We denote the set of “near equivalence” classes of domains by

$$\pi'_2(\mathbf{x}_\epsilon^{\dot{ij}_0 < \dot{ij}_n}, \mathbf{x}_\epsilon^{\dot{ij}_{n-1} < \dot{ij}_n}, \dots, \mathbf{x}_\epsilon^{\dot{ij}_0 < \dot{ij}_1}).$$

Definition 3.34. There is an obvious map

$$\phi_\epsilon: \pi_2(\mathbf{x}_\epsilon^{\dot{ij}_0 < \dot{ij}_n}, \mathbf{x}_\epsilon^{\dot{ij}_{n-1} < \dot{ij}_n}, \dots, \mathbf{x}_\epsilon^{\dot{ij}_0 < \dot{ij}_1}) \rightarrow \pi_2(\mathbf{x}_0^{i_{s_0} < i_{s_{n-k}}}, \mathbf{x}_0^{i_{s_{n-k}} < i_{s_{n+1-k}}}, \dots, \mathbf{x}_0^{i_{s_0} < i_{s_1}})$$

gotten by taking the multiplicities away from the isotopy region. More precisely, for each attaching circle β_k^i in β^i , choose two basepoints, one on either side of β_k^i , but far enough away so as to be disjoint from all translates β_ϵ^i for all $\epsilon > 0$ sufficiently small. Then $\phi_\epsilon(B_\epsilon)$ is the unique domain with the same multiplicities as B_ϵ at all of these points, and at the basepoint z . We call $B_\epsilon \in \phi_\epsilon^{-1}(B')$ an approximation to B' .

Lemma 3.35. The map ϕ_ϵ descends to a bijection

$$\pi'_2(\mathbf{x}_\epsilon^{\dot{ij}_0 < \dot{ij}_n}, \mathbf{x}_\epsilon^{\dot{ij}_{n-1} < \dot{ij}_n}, \dots, \mathbf{x}_\epsilon^{\dot{ij}_0 < \dot{ij}_1}) \xrightarrow{\cong} \pi_2(\mathbf{x}_0^{i_{s_0} < i_{s_{n-k}}}, \mathbf{x}_0^{i_{s_{n-k}} < i_{s_{n+1-k}}}, \dots, \mathbf{x}_0^{i_{s_0} < i_{s_1}}).$$

Also, two nearly equivalent homotopy classes have the same index and, for an appropriate choice of symplectic form, the same energy.

Proof. The first part of the statement follows readily from the identification

$$\pi_2(\mathbf{x}_0^{i_0 < i_n}, \mathbf{x}_0^{i_{s_{n-k}} < i_{s_{n+1-k}}}, \dots, \mathbf{x}_0^{i_{s_0} < i_{s_1}}) \cong \mathbb{Z} \oplus \text{Ker} \left(\bigoplus_{s=0}^{n-1} \text{Span}([\beta^{i_s < i_{s+1}}]) \rightarrow H_1(\Sigma; \mathbb{Z}) \right)$$

(see for example [OSz04b, Proposition 8.2]). The fact that nearly equivalent homotopy classes have the same index and energy follows from the fact that periodic domains appearing in the equivalence relation have Maslov index zero and, with respect to a symplectic form constructed by Perutz [Per08, Section 7] which agrees with the area form [OSz04b, Lemma 4.12] away from the diagonal, zero energy. \square

The dimensions of the moduli spaces \mathcal{M}^{B_ϵ} and $\mathcal{M}^{B'}$ are related by the following:

Lemma 3.36. Fix a subsequence $\dot{ij}_0 < \dots < \dot{ij}_n$ of $\mathbb{I} \times \mathbb{J}$, and let $k = R(i_0, \dots, i_n)$ be the number of repeated entries in the sequence $i_0 \leq \dots \leq i_n$. Fix a homology class

$$B' \in \pi_2(\mathbf{x}_0^{i_{s_0} < i_{s_{n-k}}}, \mathbf{x}_0^{i_{s_{n-k-1}} < i_{s_{n-k}}}, \dots, \mathbf{x}_0^{i_{s_0} < i_{s_1}})$$

and let $B_\epsilon \in \pi_2(\mathbf{x}_\epsilon^{\dot{ij}_0 < \dot{ij}_n}, \mathbf{x}_\epsilon^{\dot{ij}_{n-1} < \dot{ij}_n}, \dots, \mathbf{x}_\epsilon^{\dot{ij}_0 < \dot{ij}_1})$ be an approximation to it. Then,

$$\text{ind}(B_\epsilon) = \text{ind}(B').$$

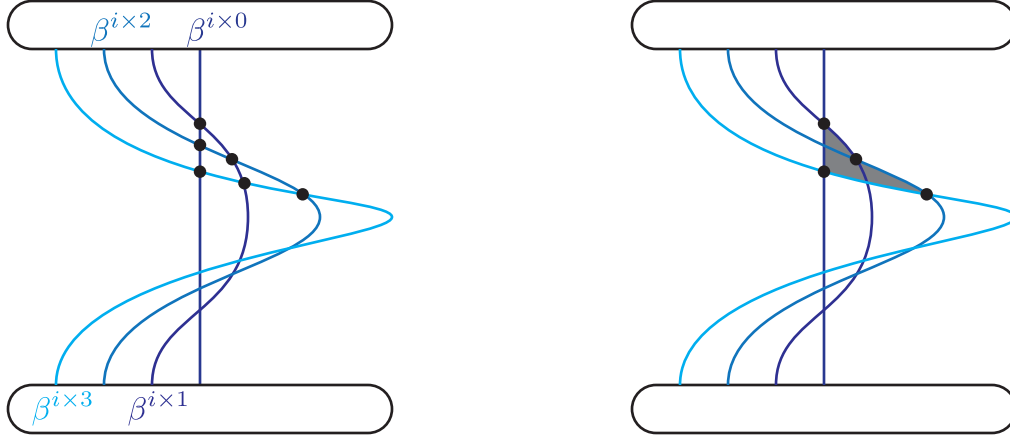


FIGURE 7. **A close approximation.** Left: Hamiltonian perturbations of $\beta^{i,0}$ giving an efficient approximation. If the distance ϵ between the $\beta^{i,j}$'s is sufficiently small then this approximation is close (Definition 3.37). (The generators $\Theta^{i \times j < i \times j'}$ are marked.) Right: the domain B_1 from Lemma 3.50.

So, for a generic choice of admissible almost-complex structure,

$$\dim(\mathcal{M}^{B_\epsilon}) - \dim(\mathcal{M}^{B'}) = k$$

except if either:

- (1) B' is the trivial homology class of bigons, i.e., the domain corresponding to B' is 0 and $k = n - 1$. In this case, $\dim(\mathcal{M}^{B'}) = 0$ and $\dim(\mathcal{M}^{B_\epsilon}) = k - 1 = n - 2$.
- (2) B' is the trivial homology class of 1-gons, i.e., the domain corresponding to B' is 0 and $k = n$. In this case, $\dim(\mathcal{M}^{B'}) = 0$ and $\dim \mathcal{M}^{B_\epsilon} = \max\{n - 2, 0\}$.

Proof. It suffices to consider the case $k = 1$: the general case follows by induction. Relabeling, let $\{\beta^i\}_{i=1}^n$ be the curves in the boundary of B_ϵ and $\{\gamma^i\}_{i=1}^{n+1}$ be the curves in the boundary of B' . Invariance properties of the index imply that $\text{ind}(B_\epsilon) - \text{ind}(B')$ is independent of the choice of Hamiltonian isotopy used to define the efficient approximation. In particular, we may assume that there is a j so that:

- $\gamma^i = \beta^i$ for $i < j$
- $\gamma^i = \beta^{i-1}$ for $i > j$.
- γ^j intersects $\beta^{j-1} = \gamma^{j-1}$ in two points, as in Figure 8.

In this case, the homotopy class B_ϵ decomposes as a juxtaposition of B' and a triangle T . (Again, see Figure 8.) Now, $\text{ind}(T) = 0$, so by additivity of the index,

$$\text{ind}(B_\epsilon) = \text{ind}(B') + \text{ind}(T) = \text{ind}(B').$$

The expected dimension of $\mathcal{M}^{B'}$ is $\text{ind}(B') + n - 3$, and the expected dimension of \mathcal{M}^{B_ϵ} is $\text{ind}(B_\epsilon) + n + 1 - 3 = \text{ind}(B') + n + 1 - 3$. Thus:

- If $\mathcal{M}^{B'}$ and \mathcal{M}^{B_ϵ} are transversally cut out then $\dim \mathcal{M}^{B_\epsilon} - \dim \mathcal{M}^{B'} = 1 = k$. This happens for generic J except if B' is the homology class of a constant map (which must be a bigon or 1-gon).
- If B' is the homology class of a constant bigon then $\dim \mathcal{M}^{B'} = 0$ and $\dim \mathcal{M}^{B_\epsilon} = 0 = k - 1$.

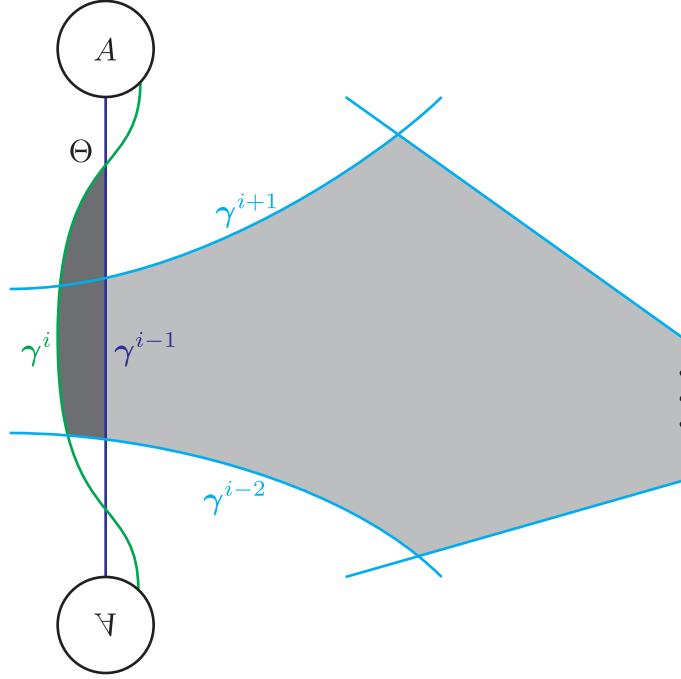


FIGURE 8. **A particular close approximation.** The domain B_ϵ is shaded; the domain $B' \subset B_\epsilon$ is lightly shaded and $T \subset B_\epsilon$ is darkly shaded.

- If B' is the homology class of a constant 1-gon then $\dim \mathcal{M}^{B'} = 0$ and $\dim \mathcal{M}^{B_\epsilon} = 0$.

This completes the proof. \square

Definition 3.37. An efficient approximation $\{\beta^{i \times j}\}_{i \times j \in \mathbb{I} \times \mathbb{J}}$ is said to be close if the following conditions hold:

- (cl-1) Each generator $\mathbf{x}^{i_0 < i_1} \in \mathfrak{S}(\beta^{i_0}, \beta^{i_1})$ has a canonical corresponding “nearest-point” generator $\mathbf{x}^{i_0 \times j_0 < i_1 \times j_1} \in \mathfrak{S}(\beta^{i_0 \times j_0}, \beta^{i_1 \times j_1})$, and the corresponding “nearest-point map” induces an isomorphism of chain complexes.

$$\widehat{CF}(\beta^{i_0}, \beta^{i_1}, z) \rightarrow \widehat{CF}(\beta^{i_0 \times j_0}, \beta^{i_1 \times j_1}, z).$$

- (cl-2) For each $i_0 < i_1$ and $j_0 < j_1$, the map

$$m_2(\Theta^{i_1 \times j_0 < i_1 \times j_1}, \cdot): \widehat{CF}(\beta^{i_0 \times j_0}, \beta^{i_1 \times j_0}, z) \rightarrow \widehat{CF}(\beta^{i_0 \times j_0}, \beta^{i_1 \times j_1}, z)$$

gotten by counting holomorphic triangles using $\Theta^{i_1 \times j_0 < i_1 \times j_1}$ coincides with the nearest-point map

$$\widehat{CF}(\beta^{i_0 \times j_0}, \beta^{i_1 \times j_0}, z) \rightarrow \widehat{CF}(\beta^{i_0 \times j_0}, \beta^{i_1 \times j_1}, z).$$

Similarly, the map

$$m_2(\cdot, \Theta^{i_0 \times j_0 < i_0 \times j_1}): \widehat{CF}(\beta^{i_0 \times j_1}, \beta^{i_1 \times j_1}, z) \rightarrow \widehat{CF}(\beta^{i_0 \times j_0}, \beta^{i_1 \times j_1}, z)$$

gotten by counting holomorphic triangles using $\Theta^{i_0 \times j_0 < i_0 \times j_1}$ coincides with the corresponding nearest-point map.

(cl-3) For each $i_0 \times j_0 < \dots < i_n \times j_n$ with $R(i_0, \dots, i_n) = k$, if $B_\epsilon \in \pi_2(\mathbf{x}_\epsilon^{j_0 < j_n}, \mathbf{x}_\epsilon^{j_{n-1} < j_n}, \dots, \mathbf{x}_\epsilon^{j_0 < j_1})$ approximates some $B' \in \pi_2(\mathbf{x}_0^{i_{s_0} < i_{s_n-k}}, \mathbf{x}_0^{i_{s_{n-k-1}} < i_{s_n-k}}, \dots, \mathbf{x}_0^{i_{s_0} < i_{s_1}})$ with $\text{ind}(B') < 3 - n$, then \mathcal{M}^{B_ϵ} is empty.

Note that, in light of Lemma 3.36, Condition (cl-3) is only interesting when $\text{ind}(B') = 2 - n$. The goal of the rest of this subsection is to prove the existence of close approximations.

Lemma 3.38. For $\{\beta_\epsilon^{i \times j}\}_{i \times j \in \mathbb{I} \times \mathbb{J}}$ a family of efficient approximations as above, fix a subsequence $j_0 < \dots < j_n$ of $\mathbb{I} \times \mathbb{J}$ with $R(i_0, \dots, i_n) = k$ and a homology class

$$B' \in \pi_2(\mathbf{x}_0^{i_{s_0} < i_{s_n-k}}, \mathbf{x}_0^{i_{s_{n-k-1}} < i_{s_n-k}}, \dots, \mathbf{x}_0^{i_{s_0} < i_{s_1}}).$$

Let $B_\epsilon \in \pi_2(\mathbf{x}_\epsilon^{j_0 < j_n}, \mathbf{x}_\epsilon^{j_{n-1} < j_n}, \dots, \mathbf{x}_\epsilon^{j_0 < j_1})$ be approximations to B' (in the sense of Definition 3.34). If \mathcal{M}^{B_ϵ} is non-empty for $\epsilon > 0$ arbitrarily small then $\mathcal{M}^{B'}$ is non-empty, as well.

Proof. For both conceptual and notational clarity, we give this proof in the symmetric product formulation. The result in the cylindrical formulation then follows from the tautological correspondence, Lemma 3.13.

Suppose we have curves $u_\epsilon \in \mathcal{M}^{B_\epsilon}$ for some infinite subsequence of $\epsilon > 0$ converging to zero. Since we are fixing the homotopy class B' , Lemma 3.35 gives an *a priori* energy estimate throughout the sequence. Gromov compactness implies that away from finitely many points and arcs in the source, there is a subsequence of the u_ϵ which converges in C_{loc}^∞ to a possibly degenerate pseudo-holomorphic $(n+1)$ -gon. (See Theorem 4.1.1 of [MS04] for a nice treatment of Gromov compactness. Note that the current application appears to be slightly more general than the version stated there for the following three reasons: we have not one but several Lagrangians, our almost-complex structures are parameterized by points in the source, and the Lagrangians move in the sequence. Because the proof of Theorem 4.1.1 is local in the source, the fact that we have several Lagrangians causes no additional complications. Since pseudo-holomorphic curves with variable almost-complex structures in the target give rise to pseudo-holomorphic curves in a product space, the second point also causes no additional difficulties. Finally, by taking a product with \mathbb{C} , a similar reduction takes care of the third point.)

The limiting object is a possibly degenerate pseudo-holomorphic $(n+1)$ -gon; i.e. the source of this curve is equipped with $n+1$ punctures, the *obligatory punctures*, arising as the limits of the punctures in the sources, and possibly additional punctures corresponding to where the conformal structure on the $(n+1)$ -gon degenerates, which we call *optional punctures*. By the removable singularities theorem or convergence of holomorphic curves to Lagrangian intersection points (depending on the puncture), we can extend the map continuously across all the punctures.¹ At the optional punctures, the limit can be taken carefully to get a sequence of bigons which connect pairs of optional punctures, so that the resulting nodal surface is connected.

¹As explained in the next paragraph, punctures can be either ephemeral, if the Lagrangians on the two sides of the puncture are the same, or persistent if the Lagrangians on the two sides are different. See, for instance, [MS04, Theorem 4.1.2] for the removable singularities theorem, which is relevant at the ephemeral punctures. Convergence of holomorphic curves to intersection points, which is relevant at the persistent punctures, is a special case of [Flo88, Theorem 2] or see, for instance, [WW10, Lemma 4.2.1]. In both cases, we use Perutz's result [Per08] that the symmetric product is symplectic and the Heegaard tori are Lagrangian, so that the hypotheses of the theorems are satisfied; see also the discussion around [MS04, Theorem 4.1.2].

We consider now the obligatory punctures. The s^{th} obligatory puncture marks two consecutive edges, which are mapped to the Heegaard tori associated to β^{i_s} and $\beta^{i_{s+1}}$. There are two cases, according to whether or not $i_s = i_{s+1}$. Punctures with $i_s = i_{s+1}$ we call *ephemeral punctures*, and those with $i_s \neq i_{s+1}$ we call *persistent punctures*. Suppose that a puncture p is persistent. Then, if the limiting curve u' does not take p to $\mathbf{x}_0^{i_s < i_{s+1}}$, but rather to some other $\mathbf{y}^{i_s < i_{s+1}} \in \mathfrak{S}(\beta^{i_s}, \beta^{i_{s+1}})$, we can reparameterize the sources to extract a chain of bigons connecting $\mathbf{x}_0^{i_s < i_{s+1}}$ and $\mathbf{y}^{i_s < i_{s+1}}$. We think of these bigons as forming part of the Gromov limiting curve u' .

At each ephemeral puncture, we do not apply any further reparameterization; simply apply the removable singularities theorem to extend the map smoothly across that puncture.

We now have a possibly degenerate pseudo-holomorphic $(n+1)$ -gon representing some class

$$B'' \in \pi_2(\mathbf{x}_\epsilon^{\dot{ij}_0 < \dot{ij}_n}, \mathbf{x}_\epsilon^{\dot{ij}_{n-1} < \dot{ij}_n}, \dots, \mathbf{x}_\epsilon^{\dot{ij}_0 < \dot{ij}_1}).$$

Next, we argue that, perhaps after including some more bigons in the limit u' , we have $B'' = B'$.

To this end let $\{p_i\}_{i=1}^m$ be the points appearing as in Definition 3.34. We can assume that all the curves in our sequence intersect the divisor $\{p_i\} \times \text{Sym}^{g-1}(\Sigma)$ transversally; in fact, there is some integer m_i with the property that $u_\epsilon^{-1}(p_i \times \text{Sym}^{g-1}(\Sigma))$ is a degree m_i divisor in the source of u_ϵ . We must argue that the preimage under u' of $p_i \times \text{Sym}^{g-1}(\Sigma)$ is also a degree m_i divisor in the (possibly nodal) source of u' .

Consider the sequence of divisors $u_\epsilon^{-1}(p_i \times \text{Sym}^{g-1}(\Sigma))$. Let S denote the source of u' and \bar{S} the result of filling in the punctures of S , so \bar{S} is a compact surface with boundary. We can assume, by passing to a subsequence, that the sequence $u_\epsilon^{-1}(p_i \times \text{Sym}^{g-1}(\Sigma))$ converges to a divisor in \bar{S} .

Clearly, if a subsequence of points in the divisors converge to some point p in the interior of \bar{S} , then u' maps p into $p_i \times \text{Sym}^{g-1}(\Sigma)$.

Taking the Gromov limit carefully, we can rule out the case where some subsequence in the divisors converges to a persistent puncture. If a subsequence runs off a puncture, reparameterize the surface to extract a sequence of bigons. In the limit, then, the sequence of divisors limit to points in the interior of a chain of disks attached at the persistent puncture. We can similarly eliminate a subsequence of the divisors converging to an optional puncture: by attaching bigons to the source, the subsequence will limit to the interior of one of the bigons attached at the optional puncture.

It is impossible for some subsequence in the divisors to converge to a point in the boundary of the source of u' : for in that case, we could extract a holomorphic disk which meets $p_i \times \text{Sym}^{g-1}(\Sigma)$ with boundary in one of the Heegaard tori (since the divisor $p_i \times \text{Sym}^{g-1}(\Sigma)$ is disjoint from the Heegaard tori). But any holomorphic disk with boundary contained in some Heegaard torus has the same local multiplicity at p_i as it does at z , which we assumed to be zero.

In a similar vein, it is impossible for some subsequence in the divisors to converge to a point which is an ephemeral puncture. For if there were such a subsequence, we would be able to translate back from the puncture to obtain a pseudo-holomorphic strip $[0, 1] \times \mathbb{R}$ with finite energy, satisfying the boundary conditions that $\{0\} \times \mathbb{R}$ and $\{1\} \times \mathbb{R}$ are mapped into some fixed Heegaard torus, and the normalization condition that point $s \times 0$ is obtained as a limit of points mapped into $p_i \times \text{Sym}^{g-1}(\Sigma)$. In the case where the limit point lies in the interior (i.e. $0 < s < 1$), by the local C^∞ convergence we conclude that $u(s, 0) \in p_i \times \text{Sym}^{g-1}(\Sigma)$.

(We are using the fact that there are no sphere components in the Gromov limit: non-trivial spheres have non-zero local multiplicity at z .) Thus, after removing the singularities at the two punctures on the boundary, we can view the result as a holomorphic disk with boundary in a Heegaard torus with non-zero multiplicity at p_i , a contradiction. The possibility that $s = 0$ or 1 is ruled out similarly. \square

Proposition 3.39. *Let $\{\beta^i\}_{i \in \mathbb{I}}$ be an admissible collection of attaching circles, and \mathbb{J} another partially ordered set. Then, given a generic, admissible collection of almost-complex structures, there is an $\mathbb{I} \times \mathbb{J}$ -filtered close approximation $\{\beta^{i \times j}\}_{i \times j \in \mathbb{I} \times \mathbb{J}}$ to $\{\beta^i\}_{i \in \mathbb{I}}$ (in the sense of Definitions 3.31 and 3.37).*

Proof. For each circle in β^i consider the Hamiltonian perturbations shown in Figure 7 (for some total order extending the partial order on \mathbb{I}). For any ϵ , these perturbations are efficient. Given a generic, admissible almost-complex structure, the moduli spaces of holomorphic curves in $(\Sigma, \beta^{i_0}, \beta^{i_1}, z)$ are transversally cut out. In particular, the moduli spaces do not change if we perturb β^{i_0} and β^{i_1} slightly. So, for ϵ small enough, Condition (cl-1) holds.

Inspecting the diagram, there is a small triangle giving the nearest-point map as a term in $m_2(\Theta^{i_1 \times j_0 < i_1 \times j_1}, \cdot)$. It follows from Lemma 3.38 that these are the only terms. Specifically, let B_ϵ be the domain of an index 0 triangle and suppose that B_ϵ admits a holomorphic representative for arbitrarily small ϵ . The domain B_ϵ approximates some domain of bigons B , and by Lemma 3.38 the domain B has a holomorphic representative. By Lemma 3.36, the index of B is also 0. But this implies B must be the trivial homology class. Hence, B_ϵ is supported in the isotopy region. Inspecting the diagram, this in turn implies that B_ϵ is the small triangle class already considered. So, Condition (cl-2) holds.

The argument for Condition (cl-3) is similar. Again, if B_ϵ admits a holomorphic representative for arbitrarily small ϵ then by Lemma 3.38 B' admits a holomorphic representative. But since $\text{ind}(B') < 3 - n$ this contradicts the assumption that we were working with a generic family of almost-complex structures. \square

3.5. Connected sums of chain complexes of attaching circles. To define the gluing construction from Proposition 1.4, we will form connected sums of Heegaard surfaces.

Definition 3.40. *Let $(\Sigma_1, \mathbb{I}, \{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i_1 < i_2}\}_{i_1, i_2 \in \mathbb{I}}, z)$ and $(\Sigma_2, \mathbb{J}, \{\gamma^j\}_{j \in \mathbb{J}}, \{\zeta^{j_1 < j_2}\}_{j_1, j_2 \in \mathbb{J}}, z)$ be chain complexes of attaching circles. We form a new $\mathbb{I} \times \mathbb{J}$ -filtered chain complex of attaching circles*

$$(\Sigma_1 \# \Sigma_2, \{\delta^{i \times j}\}_{i \times j \in \mathbb{I} \times \mathbb{J}}, \{\omega^{\tilde{i}_1 < \tilde{i}_2}\}_{\tilde{i}_1, \tilde{i}_2 \in \mathbb{I} \times \mathbb{J}}, z),$$

as follows. The surface $\Sigma_1 \# \Sigma_2$ is obtained by taking the connected sum of Σ_1 and Σ_2 near the basepoint z . Let $\{\beta^{i \times j}\}_{i \times j \in \mathbb{I} \times \mathbb{J}}$ be a close approximation to $\{\beta^i\}_{i \in \mathbb{I}}$ (in the sense of Definitions 3.31 and 3.37) and let $\{\gamma^{i \times j}\}_{i \times j \in \mathbb{I} \times \mathbb{J}}$ be a close approximation to $\{\gamma^j\}_{j \in \mathbb{J}}$. We then define

$$\delta^{i \times j} = \beta^{i \times j} \cup \gamma^{i \times j} \subset \Sigma_1 \# \Sigma_2.$$

Define chains

$$\omega^{\tilde{j}_0 < \tilde{j}_1} \in \widehat{CF}(\beta^{i_0 \times j_0}, \gamma^{i_1 \times j_1}, z)$$

by

$$(3.41) \quad \omega^{\tilde{j}_0 < \tilde{j}_1} = \begin{cases} \Theta^{i_0 \times j_0 < i_0 \times j_1} \otimes \zeta^{i_0 \times j_0 < i_0 \times j_1} & \text{if } i_0 = i_1 \\ \eta^{i_0 \times j_0 < i_1 \times j_0} \otimes \Theta^{i_0 \times j_0 < i_1 \times j_0} & \text{if } j_0 = j_1 \\ 0 & \text{otherwise.} \end{cases}$$

Here, $\zeta^{i_0 \times j_0 < i_0 \times j_1}$ and $\eta^{i_0 \times j_0 < i_1 \times j_0}$ are generators in Σ_2 and Σ_1 respectively gotten by applying the nearest-point map to $\zeta^{j_0 < j_1}$ and $\eta^{i_0 < i_1}$ respectively; and the right-hand-side of Equation (3.41) uses the Künneth isomorphism of chain groups (ignoring the differential)

$$(3.42) \quad \widehat{CF}(\beta^{\dot{j}_0}, \beta^{\dot{j}_1}, z) \otimes \widehat{CF}(\gamma^{\dot{j}_0}, \gamma^{\dot{j}_1}, z) \xrightarrow{\cong} \widehat{CF}(\beta^{\dot{j}_0} \cup \gamma^{\dot{j}_0}, \beta^{\dot{j}_1} \cup \gamma^{\dot{j}_1}, z).$$

The resulting $\mathbb{I} \times \mathbb{J}$ -filtered complex is called the connected sum of chain complexes of attaching circles $(\Sigma_1, \mathbb{I}, \{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i_1 < i_2}\}_{i_1, i_2 \in \mathbb{I}}, z)$ and $(\Sigma_2, \mathbb{J}, \{\gamma^j\}_{j \in \mathbb{J}}, \{\zeta^{j_1 < j_2}\}_{j_1, j_2 \in \mathbb{J}}, z)$ and denoted

$$(\Sigma_1, \{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i_1 < i_2}\}_{i_1, i_2 \in \mathbb{I}}, z) \# (\Sigma_2, \{\gamma^j\}_{j \in \mathbb{J}}, \{\zeta^{j_1 < j_2}\}_{j_1, j_2 \in \mathbb{J}}, z).$$

We would like to show that the construction from Definition 3.40 indeed gives a chain complex of attaching circles. This will involve analyzing moduli spaces of polygons in a connected sums of surfaces (Lemmas 3.45 and 3.50 below), and their limits as curves are isotoped (Proposition 3.39 above).

Before stating the lemmas we will need about polygons in connected sums, we review some topological preliminaries. Fix $\mathbb{I} \times \mathbb{J}$ -filtered chain complexes of attaching circles $\{\beta^{i \times j}\}_{i \times j \in \mathbb{I} \times \mathbb{J}}$ and $\{\gamma^{i \times j}\}_{i \times j \in \mathbb{I} \times \mathbb{J}}$ in Σ_1 and Σ_2 respectively. Every homotopy class

$$B \in \pi_2(\mathbf{x}^{\dot{j}_0 < \dot{j}_n} \otimes \mathbf{y}^{\dot{j}_0 < \dot{j}_n}, \mathbf{x}^{\dot{j}_{n-1} < \dot{j}_n} \otimes \mathbf{y}^{\dot{j}_{n-1} < \dot{j}_n}, \dots, \mathbf{x}^{\dot{j}_0 < \dot{j}_1} \otimes \mathbf{y}^{\dot{j}_0 < \dot{j}_1})$$

with $n_z(B) = 0$ has a corresponding splitting as $B = B_1 \# B_2$, where

$$B_1 \in \pi_2(\mathbf{x}^{\dot{j}_0 < \dot{j}_n}, \mathbf{x}^{\dot{j}_{n-1} < \dot{j}_n}, \dots, \mathbf{x}^{\dot{j}_0 < \dot{j}_1})$$

for the multi-diagram $(\Sigma_1, \{\beta^{i \times j}\}_{i \times j \in \mathbb{I} \times \mathbb{J}}, z)$ and

$$B_2 \in \pi_2(\mathbf{y}^{\dot{j}_0 < \dot{j}_n}, \mathbf{y}^{\dot{j}_{n-1} < \dot{j}_n}, \dots, \mathbf{y}^{\dot{j}_0 < \dot{j}_1})$$

for the multi-diagram $(\Sigma_2, \{\gamma^{i \times j}\}_{i \times j \in \mathbb{I} \times \mathbb{J}}, z)$.

Consider the forgetful map

$$(3.43) \quad \kappa_{B_1}: \mathcal{M}^{B_1} \rightarrow \text{Conf}(D_{n+1}),$$

defined by $\kappa_{B_1}(j, u) = j$, and the corresponding map $\kappa_{B_2}: \mathcal{M}^{B_2} \rightarrow \text{Conf}(D_{n+1})$.

Definition 3.44. Let $\{J_j^1\}$ and $\{J_j^2\}$ be admissible collections of almost-complex structures for Σ_1 and Σ_2 , respectively. Let $\{J_j\}$ be an admissible collection of almost-complex structures for $\Sigma = \Sigma_1 \# \Sigma_2$ which agrees with $J_j^1 \amalg J_j^2$ away from the connected sum region. We call such an admissible collection of almost-complex structures on Σ compatible with the splitting $\Sigma = \Sigma_1 \# \Sigma_2$.

Admissible collections of almost-complex structures compatible with the splitting $\Sigma = \Sigma_1 \# \Sigma_2$ clearly exist: they are easy to construct explicitly from $\{J_j^1\}$ and $\{J_j^2\}$.

Lemma 3.45. Fix admissible collections of almost-complex structures on Σ which are compatible with the splitting $\Sigma = \Sigma_1 \# \Sigma_2$. Then the moduli space $\mathcal{M}^{B_1 \# B_2}$ is the fibered product of $\kappa_{B_1}: \mathcal{M}^{B_1} \rightarrow \text{Conf}(D_{n+1})$ with $\kappa_{B_2}: \mathcal{M}^{B_2} \rightarrow \text{Conf}(D_{n+1})$. Further, if J_j^1 and J_j^2 are chosen generically then \mathcal{M}^{B_1} and \mathcal{M}^{B_2} are transversely cut out and the maps κ_{B_1} and κ_{B_2} are transverse to each other, so \mathcal{M}^B is a smooth manifold and

$$(3.46) \quad \dim(\mathcal{M}^B) = \dim(\mathcal{M}^{B_1}) + \dim(\mathcal{M}^{B_2}) - n + 2.$$

Proof. The description of $\mathcal{M}^{B_1 \# B_2}$ as a fibered product is immediate from the definitions. To wit, suppose

$$(u: S \rightarrow \Sigma \times D_{n+1}) \in \mathcal{M}^{B_1 \# B_2}.$$

So, u is J_j -holomorphic for some $j \in \text{Conf}(D_{n+1})$. Since the fibers of $\pi_{\mathbb{D}}$ are holomorphic (Condition (J-2)), the surface S decomposes as a disjoint union $S = S_1 \amalg S_2$, where $u(S_i) \subset \Sigma_i \times D_{n+1}$. The map $u|_{S_i}$ is a J_j^i -holomorphic representative of B_i . Conversely, given J_j^i -holomorphic representatives u_i of B_i ($i = 1, 2$), the map $u_1 \amalg u_2$ is a J_j -holomorphic representative of B .

The transversality statement follows along the lines of [MS04, Chapter 3] but with three additional complications:

- (1) We are working in the Lagrangian boundary case, rather than the closed case, so we have to work with slightly different Sobolev spaces; see, for instance, [Lip06, Section 3] for a review of the relevant spaces.
- (2) We are working with varying conformal structures on the source polygons. In the proof of transversality, one simply multiplies the source of the $\bar{\partial}$ -map by $\text{Conf}(D_{n+1})$. Of course, this makes it easier to achieve transversality (in a sense that can be made precise). In particular, the arguments from [MS04] go through essentially without change.
- (3) We want to ensure that κ_{B_1} is transverse to κ_{B_2} . It is not hard to see that for any choice of J_j^2 so that \mathcal{M}^{B_2} is transversely cut out we can choose a family J_j^1 so that \mathcal{M}^{B_1} is transversely cut out and $\text{ev}: \mathcal{M}^{B_1} \rightarrow \text{Conf}(D_{n+1})$ is transverse to \mathcal{M}^{B_2} , using the fact that in the configuration space for B_1 we have multiplied by $\text{Conf}(D_{n+1})$, and the (universal) $\bar{\partial}$ operator was already transverse to the 0-section before this multiplication.

With these hints, we leave the details of the transversality argument to the reader. \square

Lemma 3.45 has the following simple special case:

Lemma 3.47. *Let $(\Sigma_1, \beta^0, \beta^1, \beta^\infty, z)$ and $(\Sigma_2, \gamma^0, \gamma^1, \gamma^\infty, z)$ be admissible triples of attaching circles. Fix admissible collections of almost-complex structures on Σ which are compatible with the splitting $\Sigma = \Sigma_1 \# \Sigma_2$. Fix $B_1 \in \pi_2(\mathbf{x}^{0<2}, \mathbf{x}^{1<2}, \mathbf{x}^{0<1})$ and $B_2 \in \pi_2(\mathbf{y}^{0<2}, \mathbf{y}^{1<2}, \mathbf{y}^{0<1})$ with $\text{ind}(B_i) = 0$, and consider the connected sum*

$$B = B_1 \# B_2 \in \pi_2(\mathbf{x}^{0<2} \# \mathbf{y}^{0<2}, \mathbf{x}^{1<2} \# \mathbf{y}^{1<2}, \mathbf{x}^{0<1} \# \mathbf{y}^{0<1}).$$

Then,

$$(3.48) \quad \#\mathcal{M}^{B_1 \# B_2} = \#\mathcal{M}^{B_1} \cdot \#\mathcal{M}^{B_2}.$$

In particular,

$$(3.49) \quad m_2(\mathbf{x}^{1<2} \# \mathbf{y}^{1<2}, \mathbf{x}^{0<1} \# \mathbf{y}^{0<1}) = m_2(\mathbf{x}^{1<2}, \mathbf{x}^{0<1}) \otimes m_2(\mathbf{y}^{1<2}, \mathbf{y}^{0<1}).$$

Proof. Equation (3.48) follows immediately from Lemma 3.45: the space of conformal structures $\text{Conf}(D_3)$ consists of a single point, and hence the fibered product description reduces to a simple Cartesian product. Equation (3.49) follows immediately from Equation (3.48). \square

Lemma 3.50. *Fix an admissible collection of almost-complex structures on Σ which is compatible with the splitting $\Sigma = \Sigma_1 \# \Sigma_2$. Fix some g -tuple of attaching circles β^{i_0} in Σ_1 , let \mathbb{J}*

be a partially ordered set and let $\{\beta^{i_0 \times j}\}_{j \in \mathbb{J}}$ be close approximations to β^{i_0} . Then the map

$$\kappa = \coprod_{B_1} \kappa_{B_1} : \coprod_{B_1 \in \pi_2(\Theta^{i_0 \times j_0 < i_0 \times j_n}, \Theta^{i_0 \times j_{n-1} < i_0 \times j_n}, \dots, \Theta^{i_0 \times j_0 < i_0 \times j_1})} \mathcal{M}^{B_1} \rightarrow \text{Conf}(D_{n+1})$$

has degree 1 (mod 2).

This has the following consequence. Let $\{\gamma^j\}_{j \in \mathbb{J}}$ be a \mathbb{J} -filtered admissible collection of attaching circles in Σ_2 and fix a sequence $j_0 < \dots < j_n$, generators $\mathbf{y}^{j_0 < j_1}, \dots, \mathbf{y}^{j_{n-1} < j_n}$ and $\mathbf{y}^{j_0 < j_n}$ and some $B_2 \in \pi_2(\mathbf{y}^{j_0 < j_n}, \mathbf{y}^{j_{n-1} < j_n}, \dots, \mathbf{y}^{j_0 < j_1})$ for the multi-diagram $(\Sigma_2, \{\gamma^j\}_{j \in J}, z)$. Let $\mathbf{x}^{i_0 \times j_p < i_0 \times j_q} = \Theta^{i_0 \times j_p < i_0 \times j_q} \# \mathbf{y}^{j_p < j_q}$. If $\dim(\mathcal{M}^{B_2}) = 0$ then

$$(3.51) \quad \#\mathcal{M}^{B_2} \equiv \# \left(\bigcup_{B_1 \in \pi_2(\Theta^{i_0 \times j_0 < i_0 \times j_n}, \Theta^{i_0 \times j_{n-1} < i_0 \times j_n}, \dots, \Theta^{i_0 \times j_0 < i_0 \times j_1})} \mathcal{M}^{B_1 \# B_2} \right) \pmod{2}.$$

Proof. First, consider the case of triangles ($n = 2$). The space $\text{Conf}(D_3)$ is a single point, so the statement is equivalent to the statement that the triangle map is given by

$$F(\Theta^{i_0 \times j_1 < i_0 \times j_2} \otimes \Theta^{j_0 \times j_0 < i_0 \times j_1}) = \Theta^{i_0 \times j_0 < i_0 \times j_2}.$$

This is well-known. To verify it, work first with a particular Heegaard triple—for instance, a sub-diagram of Figure 7—and then observe that the triangle count is an isotopy invariant.

Next we turn to the general case. The top-dimensional boundary of

$$\overline{\mathcal{M}} := \coprod_{B_1 \in \pi_2(\Theta^{i_0 \times j_0 < i_0 \times j_n}, \Theta^{i_0 \times j_{n-1} < i_0 \times j_n}, \dots, \Theta^{i_0 \times j_0 < i_0 \times j_1})} \overline{\mathcal{M}}^{B_1} \rightarrow \overline{\text{Conf}}(D_{n+1})$$

(where $\overline{\mathcal{M}}^{B_1}$ denotes the compactification of \mathcal{M}^{B_1}) has two kinds of points:

- Boundary points where the $(n+1)$ -gon D_{n+1} degenerates as a union of two polygons with at least 3 vertices each, and
- Boundary points where the $(n+1)$ -gon D_{n+1} does not degenerate but rather the curve splits off a bigon at one of the corners.

The map κ sends the first kind of boundary points to points in the boundary of $\overline{\text{Conf}}(D_{n+1})$. The bigons occurring in points of the second kind are counted in the differential on the complex $\widehat{CF}(\beta^{i_0 \times j}, \beta^{i_0 \times j'}, z)$. So, since the differential on $\widehat{CF}(\beta^{i_0 \times j}, \beta^{i_0 \times j'}, z)$ vanishes, boundary points of the second kind occur in pairs, both lying over the same point of $\text{Conf}(D_{n+1})$. Thus, κ defines a relative cycle in $C_{n-2}(\overline{\text{Conf}}(D_{n+1}), \partial \overline{\text{Conf}}(D_{n+1}); \mathbb{Z}/2\mathbb{Z})$ and hence κ has a well-defined degree modulo 2.

Next, consider the preimage of the corner q of $\overline{\text{Conf}}(D_{n+1})$ corresponding to the decomposition of B_1 into triangles, $B_1 = B_1^n * B_1^{n-1} * \dots * B_1^2$, where

$$B_1^k \in \pi_2(\Theta^{i_0 \times j_0 < i_0 \times j_k}, \Theta^{i_0 \times j_{k-1} < i_0 \times j_k}, \Theta^{i_0 \times j_0 < i_0 \times j_{k-1}}),$$

say. By the triangle case,

$$\bigcup_{B_1^k \in \pi_2(\Theta^{i_0 \times j_0 < i_0 \times j_k}, \Theta^{i_0 \times j_{k-1} < i_0 \times j_k}, \Theta^{i_0 \times j_0 < i_0 \times j_{k-1}})} \mathcal{M}^{B_1^k}$$

has (algebraically) one holomorphic representative. Moreover, by Oh's boundary perturbation technique [Oh96], say, this representative is transversely cut out. Thus, standard gluing techniques (see, e.g., [LOT08, Proposition 5.39] for a more detailed discussion in a similar situation) imply that near this corner q , the moduli space \mathcal{M} is modeled on (an odd number

of copies of) $[0, \epsilon)^{n-1}$, projecting to $\overline{\text{Conf}}(D_{n+1})$ by local homeomorphisms of topological manifolds with corners. In particular, this implies that

$$\bigcup_{B_1^2, \dots, B_1^n} \mathcal{M}^{B_1^n} \times \mathcal{M}^{B_1^{n-1}} \times \dots \times \mathcal{M}^{B_1^2}$$

has (algebraically) one holomorphic representative, as well. Thus, κ has degree 1 (mod 2).

Finally, Equation (3.51) now follows from Lemma 3.45. \square

The following is a precise version of Proposition 1.4;

Proposition 3.52. *If the curves $\beta^{i \times j}$ and $\gamma^{i \times j}$ in Definition 3.40 are close approximations to β^i and γ^j then the connected sum of attaching circles*

$$(\Sigma_1 \cup \Sigma_2, \mathbb{I} \times \mathbb{J}, \{\delta^{i \times j}\}_{i \times j \in \mathbb{I} \times \mathbb{J}}, \{\omega^{\dot{j}_1 < \dot{j}_2}\}_{\dot{j}_1, \dot{j}_2 \in \mathbb{I} \times \mathbb{J}}, z)$$

is an $\mathbb{I} \times \mathbb{J}$ -filtered chain complex of attaching circles (Definition 3.14).

Proof. We must show the structure equation (3.15) holds for the classes ω . In fact, we will show that most terms in the Equation (3.15) vanish identically.

Consider a sequence $\dot{j}_0 < \dots < \dot{j}_n$ of indices in $\mathbb{I} \times \mathbb{J}$. Let $k = R(i_0, \dots, i_n)$ and $\ell = R(j_0, \dots, j_n)$.

Claim 1. If $k + \ell \neq n$ then

$$(3.53) \quad m_n(\omega^{\dot{j}_{n-1} < \dot{j}_n}, \dots, \omega^{\dot{j}_0 < \dot{j}_1}) = 0.$$

Indeed, in this case, some $\omega^{i_\alpha \times j_\alpha < i_{\alpha+1} \times j_{\alpha+1}}$ is itself zero.

So, from now on, we restrict to sequences with $k + \ell = n$.

Claim 2. We have

$$(3.54) \quad m_n(\omega^{\dot{j}_{n-1} < \dot{j}_n}, \dots, \omega^{\dot{j}_0 < \dot{j}_1}) = 0$$

unless one of:

- (1) $n = 2$ and $k = \ell = 1$,
- (2) $k = n$ and $\ell = 0$, or
- (3) $k = 0$ and $\ell = n$.

To see this, fix for each $m = 0, \dots, n-1$ a generator $\mathbf{x}^m \otimes \mathbf{y}^m \in \widehat{CF}(\beta^{\dot{j}_m}, \beta^{\dot{j}_{m+1}}, z)$ which is a component (summand) of the chain $\omega^{\dot{j}_m, \dot{j}_{m+1}}$. Fix also a homotopy class of $n+1$ -gons $B \in \pi_2(\mathbf{x}^n \otimes \mathbf{y}^n, \dots, \mathbf{x}^0 \otimes \mathbf{y}^0)$ with $n_z(B) = 0$ and

$$(3.55) \quad \dim(\mathcal{M}^B) = 0.$$

We can decompose $B = B_1 \# B_2$, where $B_1 \in \pi_2(\mathbf{x}^n, \dots, \mathbf{x}^0)$ and $B_2 \in \pi_2(\mathbf{y}^n, \dots, \mathbf{y}^0)$. The homotopy class B_1 is an approximation to some $B'_1 \in \pi_2(\mathbf{x}_0^{i_{s_{n+1-k}}}, \dots, \mathbf{x}_0^{i_{s_1}})$ (in the Heegaard tuple $(\Sigma, \beta^{i_{s_0}}, \dots, \beta^{i_{s_{n-k+1}}}, z)$). By Lemma 3.36,

$$(3.56) \quad \dim(\mathcal{M}^{B_1}) - \dim(\mathcal{M}^{B'_1}) = k$$

unless $k \in \{n-1, n\}$ and B'_1 is trivial. Similarly, B_2 is an approximation to a homotopy class B'_2 of $(n+1-\ell)$ -gons with

$$(3.57) \quad \dim(\mathcal{M}^{B_2}) - \dim(\mathcal{M}^{B'_2}) = \ell$$

unless $\ell \in \{n-1, n\}$ and B'_2 is trivial. Combining Equations (3.55), (3.46), (3.56), (3.57), and the fact that $n = k + \ell$, we conclude that

$$(3.58) \quad \dim(\mathcal{M}^{B'_1}) + \dim(\mathcal{M}^{B'_2}) < 0$$

unless one of:

- (1) Both B'_1 and B'_2 are constant bigons. In particular, this gives $k = n - 1$, $\ell = n - 1$. So, since $k + \ell = n$, $k = \ell = 1$.
- (2) B'_1 is a constant 1-gon. Then, in particular, $k = n$ and $\ell = 0$.
- (3) B'_2 is a constant 1-gon. Then, in particular, $\ell = n$ and $k = 0$.

If one of $\dim(\mathcal{M}^{B'_1})$ or $\dim(\mathcal{M}^{B'_2})$ is negative, Property (cl-3) ensures that \mathcal{M}^{B_1} or \mathcal{M}^{B_2} (respectively) is empty, and hence, by Lemma 3.45 \mathcal{M}^B is empty. So, for \mathcal{M}^B to be non-empty, one of cases (1), (2) or (3) must occur.

Claim 3. When $n = 2$ and $k = \ell = 1$, terms in Formula (3.15) cancel in pairs.

To see this, observe that

$$\begin{aligned}
 m_2(\omega^{i_1 \times j_0 < i_1 \times j_1}, \omega^{i_0 \times j_0 < i_1 \times j_0}) \\
 &= m_2(\Theta^{i_1 \times j_0 < i_1 \times j_1} \# \zeta^{i_1 \times j_0 < i_1 \times j_1}, \eta^{i_0 \times j_0 < i_1 \times j_0} \# \Theta^{i_0 \times j_0 < i_1 \times j_1}) \\
 &= m_2(\Theta^{i_1 \times j_0 < i_1 \times j_1}, \eta^{i_0 \times j_0 < i_1 \times j_0}) \otimes m_2(\zeta^{i_1 \times j_0 < i_1 \times j_1}, \Theta^{i_0 \times j_0 < i_1 \times j_1}) \\
 &= \eta^{i_0 \times j_0 < i_1 \times j_1} \otimes \zeta^{i_0 \times j_0 < i_1 \times j_1},
 \end{aligned}$$

by the definition of ω , Lemma 3.47 (Equation (3.49)), and Property (cl-2) of the close approximation, in turn. The same argument shows that

$$m_2(\omega^{i_0 \times j_1 < i_1 \times j_1}, \omega^{i_0 \times j_0 < i_0 \times j_1}) = \eta^{i_0 \times j_0 < i_1 \times j_1} \otimes \zeta^{i_0 \times j_0 < i_1 \times j_1}.$$

This proves the claim.

Claim 4. The terms in Formula (3.15) with $k = n$ (and $\ell = 0$) cancel with each other.

Indeed, if $k = n$ then Lemma 3.50 applies: $B = B_1$ from Lemma 3.50, and that lemma ensures that $\#\mathcal{M}^{B_1 \# B_2} = \#\mathcal{M}^{B_2}$. Thus, Equation (3.15) for the ω is a consequence of the corresponding condition on the ζ : for any fixed $i_0 \in \mathbb{I}$ and $j < j' \in \mathbb{J}$,

$$\begin{aligned}
 (3.59) \quad &\sum_{j=j_0 < j_1 < \dots < j_{n-1} < j_n=j'} m_n(\omega^{i_0 \times j_{n-1} < i_0 \times j_n}, \dots, \omega^{i_0 \times j_0 < i_0 \times j_1}) \\
 &= \sum_{j=j_0 < j_1 < \dots < j_{n-1} < j_n=j'} m_n(\zeta^{j_{n-1} < j_n}, \dots, \zeta^{j_0 < j_1}) \\
 &= 0.
 \end{aligned}$$

Claim 5. The terms in Formula (3.15) with $\ell = n$ (and $k = 0$) cancel with each other. This follows from the same argument used to prove Claim 4, with the two sides of the diagram reversed.

The five claims account for all of the terms in Formula (3.15), so Formula (3.15) holds and the proposition is proved. \square

3.6. The chain complex for a link. A particular chain complex of attaching circles is constructed in [OSz05], though without using this terminology.

Let L be a c -component link in a three-manifold. A Heegaard diagram subordinate to L in Y is a pointed Heegaard diagram

$$(\Sigma, \alpha = \{\alpha_1, \dots, \alpha_g\}, \beta = \{\beta_1^\infty, \dots, \beta_c^\infty, \beta_{c+1}, \dots, \beta_g\}, z)$$

with the property that the components of L are boundary-parallel circles in the β -handlebody, the attaching disk of β_k^∞ meets the k^{th} component of L transversally in one point if $k = 1, \dots, c$, and these are the only intersection points of the β -attaching disks and L .

A tuple of framings for L can be specified by curves $\{\beta_k^0\}_{k=1}^c$, so that $\beta_k^0 \cap \beta_\ell = \emptyset$ for $k = c+1, \dots, g$ and $\beta_k^0 \cap \beta_\ell^\infty = \emptyset$ unless $k = \ell$, in which case the two curves meet transversally in one point. For $k = 1, \dots, c$, let β_k^1 be a standard resolution of $\beta_k^\infty \cup \beta_k^0$, as in Figure 3. With these choices, we have a collection of attaching circles indexed by $\mathbb{I} = \{0, 1, \infty\}^c$. Given a sequence $i_0 < \dots < i_n$ in \mathbb{I} we can consider the Heegaard multi-diagram $(\Sigma, \alpha, \beta^{i_0}, \dots, \beta^{i_n}, z)$.

We perturb the β^{i_j} 's to make this multi-diagram admissible, in an efficient way, i.e., so that the differential on $\widehat{CF}(\beta^{i_k}, \beta^{i_{k+1}}, z)$ vanishes identically. (Any periodic domain can be written as a sum of doubly- and triply-periodic domains. The doubly-periodic domains have zero area by the above construction. The triply periodic domains can be arranged to have zero area as well, see for example Figure 3.) If i_k and i_{k+1} are consecutive (i.e., i_k and i_{k+1} differ in exactly one coordinate) then let $\Theta^{i_k < i_{k+1}} \in \widehat{CF}(\beta^{i_k}, \beta^{i_{k+1}}, z)$ be the unique top-dimensional generator. Otherwise, let $\Theta^{i_k < i_{k+1}} = 0$.

Definition 3.60. *The chain complex of attaching circles*

$$(\mathbb{I} = \{0, 1, \infty\}^c, \{\beta^i\}_{i \in \mathbb{I}}, \{\Theta^{i < i'}\}_{i, i' \in \mathbb{I}}, z)$$

is called the chain complex of framing changes. If we further specify an additional g -tuple of attaching circles α , and let Y be specified by the Heegaard diagram $(\Sigma, \alpha, \beta^\infty, z)$, and $L \subset Y$ be the corresponding framed link, then we say that $(\mathbb{I} = \{0, 1, \infty\}^c, \{\beta^i\}_{i \in \mathbb{I}}, \{\Theta^{i < i'}\}_{i, i' \in \mathbb{I}}, z)$ is the chain complex of framing changes on the link $L \subset Y$ specified by α .

Definitions 3.18 and 3.60 make

$$\bigoplus_{i \in \{0, 1, \infty\}^c} \widehat{CF}(\alpha, \{\beta^i\}_{i \in \{0, 1, \infty\}^c}, z)$$

into a chain complex. The paper [OSz05] also makes this vector space into a chain complex, denoted X , by counting pseudo-holomorphic polygons [OSz05, Section 4.2].

Lemma 3.61. *The chain complex X is the filtered complex associated in Definition 3.18 to α and the chain complex associated to framing changes on the link (Definition 3.60).*

Proof. The complex X from [OSz05, Section 4.2] is $\bigoplus_{i \in \mathbb{I}} \widehat{CF}(\alpha, \beta^i, z)$ with differential given by

$$D(\xi) = \sum_{i_1 < \dots < i_k \text{ consecutive}} m_k(\Theta^{i_{k-1} < i_k}, \dots, \Theta^{i_1 < i_2}, \xi).$$

This is exactly the complex from Definition 3.18. □

4. POLYGON COUNTING IN BORDERED MANIFOLDS

The present section contains a generalization of some of the material from Section 3 to the bordered setting. In Subsection 4.1, we introduce bordered multi-diagrams (generalizing the earlier Heegaard multi-diagrams). In Subsection 4.2, we consider moduli spaces of pseudo-holomorphic polygons in bordered multi-diagrams. Counting points in these moduli spaces gives the maps generalizing the pseudo-holomorphic polygon counts considered earlier. The algebra of these holomorphic curve counts is described in Subsection 4.3. In Subsection 4.4, it is shown that an \mathbb{I} -filtered chain complex of attaching circles, together with a set of bordered attaching curves, gives rise to a filtered A_∞ -module over the algebra associated to a pointed matched circle. These results, Proposition 4.27 (the Type A version) and Proposition 4.29 (the Type D version), can be viewed as bordered analogues of the filtered chain complexes

constructed in Proposition 3.19. The filtered type A modules and type D modules will appear in the statement of the pairing theorem for polygons in Section 5. Subsection 4.5 describes the further generalization to bordered Heegaard diagrams with two boundary components (in the spirit of [LOT15]). These generalizations appear in the statement of a pairing theorem used to prove the main theorem of this paper, Theorem 2.

4.1. Bordered multi-diagrams. If Σ is a surface with genus g and a single boundary component, one can consider g -tuples of attaching circles as in Definition 3.1. There is another kind of tuple of curves which is natural in the bordered case:

Definition 4.1. *Let Σ be a compact, oriented surface with one boundary component. Fix a pointed matched circle \mathcal{Z} consisting of $2k$ pairs of points $\mathbf{a} \subset S^1 = \partial\Sigma$. A complete set of bordered attaching curves compatible with \mathcal{Z} is a collection $\alpha = \{\alpha_1, \dots, \alpha_{g+k}\}$ of curves in Σ such that:*

- *The curves $\alpha_i \in \alpha$ are pairwise disjoint.*
- *$\alpha \cap \partial\Sigma = \partial\alpha = \mathcal{Z}$. We sometimes abbreviate this condition as $\partial(\Sigma, \alpha) = \mathcal{Z}$.*
- *The relative cycles $\{[\alpha_i]\}_{i=1}^{g+2k}$, where $[\alpha_i] \in H_1(\Sigma, \partial\Sigma)$, are linearly independent.*

When considering holomorphic curves, we will attach a cylindrical end to $\partial\Sigma$. We will still denote the result by Σ , and hope that this will not cause confusion.

Definition 4.2. *Let α be a complete set of bordered attaching curves in Σ in the sense of Definition 4.1 (compatible with some \mathcal{Z}), and let $\{\beta^i\}_{i=1}^n$ be an n -tuple of complete sets of attaching circles (in the sense of Definition 3.1). We call the data $(\Sigma, \alpha, \beta^1, \dots, \beta^n, z)$ a bordered multi-diagram.*

A generalized multi-periodic domain is a relative homology class $B \in H_2(\Sigma, \alpha \cup \beta^1 \cup \dots \cup \beta^n \cup \partial\Sigma)$ whose boundary ∂B , viewed as an element of

$$H_1(\alpha \cup \beta^1 \cup \dots \cup \beta^n \cup \partial\Sigma, \partial\Sigma) \cong H_1(\alpha \cup \beta^1 \cup \dots \cup \beta^n, \mathbf{a})$$

is contained in the image of the inclusion

$$H_1(\alpha, \mathbf{a}) \oplus H_1(\beta^1) \oplus \dots \oplus H_1(\beta^n) \rightarrow H_1(\alpha \cup \beta^1 \cup \dots \cup \beta^n, \mathbf{a}).$$

A generalized multi-periodic domain P has a local multiplicity $n_x(P)$ at any point $x \in \Sigma \setminus (\alpha \cup \beta^1 \cup \dots \cup \beta^n)$. A multi-periodic domain is one whose local multiplicity at (the region adjacent to) the point z vanishes.

A multi-periodic domain is called provincial if all of its local multiplicities near $\partial\Sigma$ vanish; equivalently, if it has trivial boundary in $H_0(\mathbf{a})$.

We say that $(\Sigma, \alpha, \{\beta^i\}_{i \in \mathbb{I}}, z)$ is admissible if any non-zero multi-periodic domain has both positive and negative local multiplicities. The diagram $(\Sigma, \alpha, \{\beta^i\}_{i \in \mathbb{I}}, z)$ is called provincially admissible if any non-zero provincial multi-periodic domain has both positive and negative local multiplicities.

Lemma 4.3. *If $\{\beta^i\}_{i \in \mathbb{I}}$ is an \mathbb{I} -filtered, admissible collection of attaching circles (in the sense of Definition 3.2), and α is a complete set of bordered attaching curves compatible with some pointed matched circle \mathcal{Z} on $\partial\Sigma$, then we can always find another complete set of bordered attaching curves α' compatible with \mathcal{Z} so that*

- *$(\Sigma, \alpha', \{\beta^i\}_{i \in \mathbb{I}}, z)$ is admissible.*
- *α' is isotopic to α .*

Proof. This follows by winding transversely to the α curves, as in the case of bigons [OSz04b, Lemma 5.4]. (The corresponding result for bordered Heegaard diagrams (i.e., $|\mathbb{I}| = 1$), is [LOT08, Proposition 4.25].) \square

4.2. Moduli spaces of polygons in bordered manifolds. As discussed in Section 3, there are polygon counts defined in Heegaard Floer homology which satisfy the A_∞ relations. The goal of this subsection and the next is to generalize these polygon maps to the bordered context. Suppose that $(\Sigma, \alpha, \{\beta^i\}_{i \in \mathbb{I}}, z)$ is an admissible multi-diagram. We will define maps

$${}_n m_k: \widehat{CF}(\beta^{i_{n-1}}, \beta^{i_n}, z) \otimes \cdots \otimes \widehat{CF}(\beta^{i_1}, \beta^{i_2}, z) \otimes \widehat{CFA}(\alpha, \beta^{i_1}) \otimes \mathcal{A}^{\otimes k} \rightarrow \widehat{CFA}(\alpha, \beta^{i_n})$$

by combining the definition of type A modules from [LOT08, Chapter 7] with the usual polygon counts.

In this subsection, we set up the relevant moduli spaces. As is usual in the cylindrical setting, this is a two-step process. First we introduce moduli spaces of polygons with a fixed source. The expected dimension of these moduli spaces depends on the Euler characteristic of the source. We then show that for embedded polygons, the Euler characteristic of the source is determined by the homology class. (This is an extension of Sarkar's index formula for polygons [Sar11].) In the next section, we will count moduli spaces of rigid, embedded polygons to define the polygon maps.

To start, attach a cylindrical end $S^1 \times [0, \infty)$ to $\partial\Sigma$. We will denote the result by Σ , as well; this should not cause confusion.

We generalize Definition 3.5 to the bordered context:

Definition 4.4. *An admissible collection of almost-complex structures is a choice of smooth family $\{J_j\}_{j \in \text{Conf}(D_n)}$ of almost-complex structures on $\Sigma \times D_n$ for each $n \geq 3$, satisfying all the conditions in Definition 3.5, and the following further condition:*

- *over the cylindrical end $S^1 \times [0, \infty)$ of Σ , the complex structure J_j splits as a product $j_0 \times j$, where j_0 is a standard cylindrical complex structure on $S^1 \times [0, \infty)$.*

Definition 4.5. *Let $\{J_j\}_{j \in \text{Conf}(D_{n+1})}$ be an admissible collection of almost-complex structures. Fix a complete set of bordered attaching curves α compatible with \mathcal{Z} (Definition 4.1), and a further collection of n complete sets of attaching circles β^1, \dots, β^n (Definition 3.1). Fix generators $\mathbf{x}^k \in \mathfrak{S}(\beta^k, \beta^{k+1})$ for $k = 1 \dots, n-1$, as well as $\mathbf{x}^0 \in \mathfrak{S}(\alpha, \beta^1)$, $\mathbf{x}^n \in \mathfrak{S}(\alpha, \beta^n)$, and consider maps*

$$(4.6) \quad u: (S, \partial S) \rightarrow (\Sigma \times D_{n+1}, (\alpha \times e_0) \cup (\beta^1 \times e_1) \cup \cdots \cup (\beta^n \times e_n))$$

where S is a punctured Riemann surface and D_{n+1} is equipped with a set of points $q_i \in \partial D_{n+1}$ for $i = 1, \dots, k$, with the following properties:

- (c-0) *The projection $\pi_\Sigma \circ u: S \rightarrow \Sigma$ has degree 0 at the region adjacent to the basepoint z .*
- (c-1) *The punctures of S are mapped to the punctures $\{p_{i,i+1}\}_{i=0}^\ell \cup \{q_i\}_{i=1}^k$ of $D_{n+1} \setminus \{q_i\}_{i=1}^k$.*
- (c-2) *The curve u is asymptotic to $\mathbf{x}^i \times \{p_{i,i+1}\}$ at the preimage of the puncture $p_{i,i+1}$.*
- (c-3) *The curve u is asymptotic to $\rho_i \times \{q_i\}$ at the punctures above q_i , for some set of Reeb chords ρ_i (in $Z \setminus \{z\}$ with endpoints in \mathbf{a}).*
- (c-4) *At each point $q \in (e_0 \setminus \{q_i\}_{i=1}^\ell)$, the g points $(\pi_\Sigma \circ u)((\pi_{\mathbb{D}} \circ u)^{-1}(q))$ lie in g distinct α -curves.*

The set of such u decomposes into homology classes, denoted $\pi_2(\mathbf{x}^n, \mathbf{x}^{n-1}, \dots, \mathbf{x}^0; \rho_1, \dots, \rho_m)$. For fixed $B \in \pi_2(\mathbf{x}^n, \mathbf{x}^{n-1}, \dots, \mathbf{x}^0; \rho_1, \dots, \rho_m)$, let

$$\mathcal{M}^{B,S} = \mathcal{M}^B(\mathbf{x}^n, \dots, \mathbf{x}^0; \rho_1, \dots, \rho_m; S)$$

denote the moduli space pairs (j, u) where $j \in \text{Conf}(D_{n+1})$ and u is a J_j -holomorphic representative of $B \in \pi_2(\mathbf{x}^n, \dots, \mathbf{x}^0; \boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_m)$.

Condition (c-4) can be formulated as a combinatorial condition on the $(\mathbf{x}, \boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_m)$: it is the *strong boundary monotonicity* of [LOT08, Section 5.2]. It is equivalent to the algebraic condition that $\mathbf{x} \otimes a(\boldsymbol{\rho}_1) \otimes \dots \otimes a(\boldsymbol{\rho}_m)$ is a non-vanishing element in $\widehat{CFA}(\boldsymbol{\alpha}, \boldsymbol{\beta}^1) \otimes \mathcal{A}(\mathcal{Z}) \otimes \dots \otimes \mathcal{A}(\mathcal{Z})$, where the tensor is taken over the ring of idempotents in $\mathcal{A}(\mathcal{Z})$; see [LOT08, Lemma 7.2].

Lemma 4.7. *The expected dimension of the moduli space $\mathcal{M}^B(\mathbf{x}^n, \dots, \mathbf{x}^0; \boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_m; S)$ is given by $\text{ind}(B, S; \boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_m) + n - 2$ where*

$$(4.8) \quad \text{ind}(B, S; \boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_m) := \left(\frac{3-n}{2} \right) g - \chi(S) + 2e(B) + m.$$

Here, g is the genus of Σ (which is also the number of elements in each \mathbf{x}^i) and $e(B)$ denotes the Euler measure of B .

Proof. This is a simple adaptation of the proof in the bigon case [LOT08, Proposition 5.8]; see also [Lip06, Section 10.2] and [Sar11] in the closed case. \square

Next, we observe that, as with bigons, embeddedness is equivalent to a condition on the Euler characteristic of S . To state the formula for $\chi(S)$ we need a little more notation:

- Given a domain B , let $\partial_i B$ denote the part of B lying along $\boldsymbol{\beta}^i$, and $\partial_0 B$ the part of B lying along $\boldsymbol{\alpha}$.
- Given two curves γ, η in Σ , with $\gamma \cap \eta$ but possibly intersecting at the endpoints of γ or η , we can define the *jittered intersection number* of γ and η , denoted $\gamma \cdot \eta$, by pushing η slightly so that the endpoints of η (respectively γ) become disjoint from γ (respectively η) in the four obvious ways, and averaging the results. See [LOT08, Section 5.7.2].
- Given a pair of Reeb chords ρ_1, ρ_2 in Z , let $L(\rho_1, \rho_2)$ be the linking number of $\partial \rho_1$ and $\partial \rho_2$, i.e., the multiplicity with which ρ_2 covers $\partial \rho_1$. (This can be a half-integer, if ρ_1 and ρ_2 share an endpoint.) Extend L bilinearly to a function $L(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$ on pairs of sets of Reeb chords. See [LOT08, Section 3.3.1 and 5.7.1].
- Given a set of Reeb chords $\boldsymbol{\rho}$, let $\iota(\boldsymbol{\rho}) = -\frac{|\boldsymbol{\rho}|}{2} - \sum_{\{\rho_1, \rho_2\} \subset \boldsymbol{\rho}} |L(\rho_1, \rho_2)|$. See [LOT08, Section 5.7.1].

Proposition 4.9. *Suppose that $u \in \mathcal{M}^B(\mathbf{x}^n, \dots, \mathbf{x}^0; \boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_m; S)$ is an embedded holomorphic $(n+1)$ -gon. Then*

$$(4.10) \quad \begin{aligned} \chi(S) = & g + e(B) - n_{\mathbf{x}^0}(B) - n_{\mathbf{x}^n}(B) \\ & - \sum_{n \geq j > \ell \geq 1} \partial_j(B) \cdot \partial_\ell(B) - \sum_i \iota(\boldsymbol{\rho}_i) - \sum_{i < j} L(\boldsymbol{\rho}_i, \boldsymbol{\rho}_j) \end{aligned}$$

$$(4.11) \quad \begin{aligned} \text{ind}(B; S; \boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_m) = & e(B) + n_{\mathbf{x}^0}(B) + n_{\mathbf{x}^n}(B) - \left(\frac{n-1}{2} \right) g \\ & + \sum_{n \geq j > \ell \geq 1} \partial_j(B) \cdot \partial_\ell(B) + m + \sum_i \iota(\boldsymbol{\rho}_i) + \sum_{i < j} L(\boldsymbol{\rho}_i, \boldsymbol{\rho}_j). \end{aligned}$$

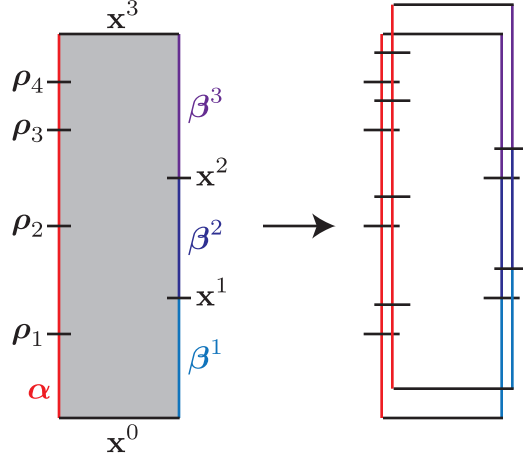


FIGURE 9. **Computing the embedded index for polygons.** The polygon shown is a 4-gon (rectangle).

Moreover, if $\mathcal{M}^B(\mathbf{x}^n, \dots, \mathbf{x}^0; \rho_1, \dots, \rho_m; S)$ contains a non-embedded holomorphic polygon then

$$\begin{aligned} \text{ind}(B; S; \rho_1, \dots, \rho_m) \leq & \left[e(B) + n_{\mathbf{x}^0}(B) + n_{\mathbf{x}^n}(B) - \left(\frac{n-1}{2} \right) g + \sum_{n \geq j > \ell \geq 1} \partial_j(B) \cdot \partial_\ell(B) \right. \\ & \left. + m + \sum_i \iota(\rho_i) + \sum_{i < j} L(\rho_i, \rho_j) \right] - 2. \end{aligned}$$

Proof. The formula is a combination of the embedded index formula for bigons [LOT08, Proposition 5.69] and Sarkar's index formula for n -gons [Sar11]. We prove it by imitating the proof of [LOT08, Proposition 5.69]; see also the proof of [LOT08, Proposition 6.25]. We will be brief.

We start with the formula for χ . View the disk D_{n+1} as the strip $[0, 1] \times \mathbb{R}$ with $(n-1)$ marked points on $\{1\} \times \mathbb{R}$, so that the boundary conditions α correspond to $\{0\} \times \mathbb{R}$ and \mathbf{x}^0 corresponds to $[0, 1] \times \{-\infty\}$. There is an induced \mathbb{R} -action on D_{n+1} which is holomorphic but does not preserve (all but two of) the corners; continuing to think of D_{n+1} as a strip, we will call this \mathbb{R} -action *translation*. Let $\tau_r(u)$ be the result of translating u by r units.

Given a map v between Riemann surfaces, let $\text{br}(v)$ denote the ramification degree of v . Recall that $\pi_{\mathbb{D}}: \Sigma \times D_n \rightarrow D_n$ and $\pi_\Sigma: \Sigma \times D_n \rightarrow \Sigma$ are the two projections. Let $\frac{\partial}{\partial t}$ be the vector field generated by the \mathbb{R} -action on D_n .

Viewing the \mathbf{x}^i punctures of S as right-angled corners and each Reeb chord as having two right-angled corners, we have

$$(4.12) \quad \chi(S) = e(S) + \frac{(n+1)g}{4} + \sum_i \frac{|\rho_i|}{2}.$$

By the Riemann-Hurwitz formula,

$$(4.13) \quad e(S) = e(B) - \text{br}(\pi_\Sigma \circ u).$$

By definition $\text{br}(\pi_\Sigma \circ u)$ is the number of tangencies of $\frac{\partial}{\partial t}$ to u . Taking into account that sliding Reeb chords in $\boldsymbol{\rho}_i$ past each other introduces boundary double points we have

$$(4.14) \quad \text{br}(\pi_\Sigma \circ u) = u \cdot \tau_\epsilon(u) - \sum_i \sum_{\{\rho_a, \rho_b\} \subset \boldsymbol{\rho}_i} L(\rho_a, \rho_b),$$

where $u \cdot \tau_\epsilon(u)$ denotes the intersection number—algebraic or geometric does not matter, since holomorphic curves intersect positively.

Translating farther, for R sufficiently large we have

$$(4.15) \quad u \cdot \tau_\epsilon(u) = u \cdot \tau_R(u) + \sum_{n \geq j > \ell \geq 1} \partial_j(B) \cdot \partial_\ell(B) + \frac{g(n-1)}{4} + \sum_{i < j} L(\boldsymbol{\rho}_i, \boldsymbol{\rho}_j).$$

Again, the contribution of $L(\boldsymbol{\rho}_i, \boldsymbol{\rho}_j)$ comes from Reeb chords sliding past each other; see, e.g., the proof of [LOT08, Proposition 5.69]. The contribution of $\sum_{n \geq j > \ell \geq 1} \partial_j(B) \cdot \partial_\ell(B) + \frac{g(n-1)}{4}$ comes from intersections appearing or disappearing along the boundary, where β^i and β^j intersect; see, the proof of [LOT08, Proposition 6.25].

As in the bigon case,

$$(4.16) \quad u \cdot \tau_R(u) = n_{\mathbf{x}^0}(B) + n_{\mathbf{x}^n}(B) - \frac{g}{2}.$$

Combining Equations (4.12), (4.13), (4.14), (4.15) and (4.16) gives

$$\begin{aligned} \chi(S) &= \frac{(n+1)g}{4} + \sum_i \frac{|\boldsymbol{\rho}_i|}{2} + e(B) - \left[- \sum_i \sum_{\{\rho_a, \rho_b\} \subset \boldsymbol{\rho}_i} L(\rho_a, \rho_b) + \sum_{n \geq j > \ell \geq 1} \partial_j(B) \cdot \partial_\ell(B) \right. \\ &\quad \left. + \frac{g(n-1)}{4} + \sum_{i < j} L(\boldsymbol{\rho}_i, \boldsymbol{\rho}_j) + n_{\mathbf{x}^0}(B) + n_{\mathbf{x}^n}(B) - \frac{g}{2} \right] \\ &= g + e(B) - n_{\mathbf{x}^0}(B) - n_{\mathbf{x}^n}(B) + \sum_i \frac{|\boldsymbol{\rho}_i|}{2} + \sum_i \sum_{\{\rho_a, \rho_b\} \subset \boldsymbol{\rho}_i} L(\rho_a, \rho_b) \\ &\quad - \sum_{n \geq j > \ell \geq 1} \partial_j(B) \cdot \partial_\ell(B) - \sum_{i < j} L(\boldsymbol{\rho}_i, \boldsymbol{\rho}_j) \\ &= g + e(B) - n_{\mathbf{x}^0}(B) - n_{\mathbf{x}^n}(B) - \sum_i \iota(\boldsymbol{\rho}_i) - \sum_{n \geq j > \ell \geq 1} \partial_j(B) \cdot \partial_\ell(B) - \sum_{i < j} L(\boldsymbol{\rho}_i, \boldsymbol{\rho}_j), \end{aligned}$$

as claimed.

Combining Formulas (4.8) and (4.10) gives

$$\begin{aligned}
\text{ind}(u) &= \left(\frac{3-n}{2} \right) g + 2e(B) + m \\
&\quad - \left[g + e(B) - n_{\mathbf{x}^0}(B) - n_{\mathbf{x}^n}(B) - \sum_i \iota(\boldsymbol{\rho}_i) \right. \\
&\quad \left. - \sum_{n \geq j > \ell \geq 1} \partial_j(B) \cdot \partial_\ell(B) - \sum_{i < j} L(\boldsymbol{\rho}_i, \boldsymbol{\rho}_j) \right] \\
&= \left(\frac{1-n}{2} \right) g + e(B) + m + n_{\mathbf{x}^0}(B) + n_{\mathbf{x}^n}(B) + \sum_i \iota(\boldsymbol{\rho}_i) \\
&\quad + \sum_{n \geq j > \ell \geq 1} \partial_j(B) \cdot \partial_\ell(B) + \sum_{i < j} L(\boldsymbol{\rho}_i, \boldsymbol{\rho}_j),
\end{aligned}$$

as claimed.

Finally, for non-embedded curves, each double point or equivalent singularity increases χ by 2, and consequently drops ind by 2. \square

Finally, we define the moduli spaces of embedded curves:

Definition 4.17. Let $\mathcal{M}^B = \mathcal{M}^B(\mathbf{x}^n, \dots, \mathbf{x}^0; \boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_m)$ denote the set of embedded holomorphic maps in the homology class B with asymptotics $\mathbf{x}^n, \dots, \mathbf{x}^0$ and $\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_m$; equivalently,

$$\mathcal{M}^B(\mathbf{x}^n, \dots, \mathbf{x}^0; \boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_m) := \bigcup_{\chi(S) \text{ given by (4.10)}} \mathcal{M}^B(\mathbf{x}^n, \dots, \mathbf{x}^0; \boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_m; S).$$

Proposition 4.18. For a generic, admissible family $\{J_j\}$ of almost-complex structures, the moduli spaces $\mathcal{M}^{B,S} = \mathcal{M}^B(\mathbf{x}^n, \dots, \mathbf{x}^0; \boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_m; S)$, and hence also the embedded moduli spaces $\mathcal{M}^B(\mathbf{x}^n, \dots, \mathbf{x}^0; \boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_m)$, are transversely cut out, and hence are smooth manifolds whose dimensions are given by $\text{ind}(B, S; \boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_m) + n - 2$ and $\text{ind}(B; \boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_m) + n - 2$, respectively.

Proof. This follows from standard transversality results; see, for instance, [LOT08, Proposition 5.6] for the analogous result for bigons. \square

The spaces $\mathcal{M}^{B,S}$ and $\mathcal{M}^B(\mathbf{x}^n, \dots, \mathbf{x}^0; \boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_m)$ are, of course, typically non-compact, except when they are 0-dimensional, in which case they are compact; see also Remark 3.9.

4.3. Maps induced by polygon counts.

Definition 4.19. Suppose that $(\Sigma, \boldsymbol{\alpha}, \{\boldsymbol{\beta}^i\}_{i=1}^n, z)$ is a provincially admissible multi-diagram. Define the map

$${}_n m_k: \widehat{CF}(\boldsymbol{\beta}^{n-1}, \boldsymbol{\beta}^n, z) \otimes \cdots \otimes \widehat{CF}(\boldsymbol{\beta}^1, \boldsymbol{\beta}^2, z) \otimes \widehat{CFA}(\boldsymbol{\alpha}, \boldsymbol{\beta}^1) \otimes \overbrace{\mathcal{A} \otimes \cdots \otimes \mathcal{A}}^k \rightarrow \widehat{CFA}(\boldsymbol{\alpha}, \boldsymbol{\beta}^n),$$

by extending the following formulas linearly. For fixed $\eta^i \in \mathfrak{S}(\beta^{i-1}, \beta^i)$ and sequence of sets of Reeb chords $\vec{\rho} = (\rho_1, \dots, \rho_k)$ define:

$$(4.20) \quad {}_n m_k(\eta^n, \dots, \eta^1, \mathbf{x}, a(\rho_1), \dots, a(\rho_k)) = \sum_{\substack{\mathbf{y} \in \mathfrak{S}(\alpha, \beta^n) \\ B \in \pi_2(\mathbf{y}, \eta^{n-1}, \dots, \eta^1, \mathbf{x}; \rho_1, \dots, \rho_m) \\ \text{ind}(B; \rho_1, \dots, \rho_m) = 2-n}} \# \mathcal{M}^B(\mathbf{y}, \eta^{n-1}, \dots, \eta^1, \mathbf{x}; \rho_1, \dots, \rho_k) \mathbf{y}$$

So, for example, ${}_0 m_0$ is the differential on $\widehat{CFA}(\alpha, \beta^0)$. Since the moduli spaces being counted are 0-dimensional, they are compact and hence finite. Provincial admissibility implies that the sum defining ${}_n m_k$ is also finite (compare [LOT08, Proposition 4.28]).

Admissibility guarantees that ${}_n m_k = 0$ for a fixed multi-diagram and all sufficiently large k (compare [LOT08, Proposition 4.29]).

The polygon counts satisfy an A_∞ relation. Before stating this relation, note that \mathcal{A} is a dg algebra, so $\mu_{i-j+1} = 0$ unless $j = i$ or $j = i - 1$.

Proposition 4.21. *The polygon counts defined above satisfy the following A_∞ relation:*

$$\begin{aligned} 0 = & \sum_{0 \leq i \leq k; 0 \leq p \leq n} {}_{n-p} m_{k-i}(\eta^n, \dots, \eta^{p+1}, {}_p m_i(\eta^p, \dots, \eta^1, \mathbf{x}, a_1, \dots, a_i), a_{i+1}, \dots, a_k) \\ & + \sum_{1 \leq p < q \leq n} {}_{n-q+p} m_k(\eta^n, \dots, \eta^{q+1}, m_{q-p+1}(\eta^q, \dots, \eta^p), \eta^{p-1}, \dots, \eta^1, \mathbf{x}, a_1, \dots, a_k) \\ & + \sum_{1 \leq i \leq j \leq n} {}_n m_{k-j+i}(\eta^n, \dots, \eta^1, \mathbf{x}, a_1, \dots, \mu_{i-j+1}(a_i, \dots, a_j), \dots, a_k), \end{aligned}$$

for any $\mathbf{x} \in \mathfrak{S}(\alpha, \beta^1)$, $\eta^i \in \widehat{CF}(\beta^i, \beta^{i+1}, z)$, $a_i \in \mathcal{A}(\mathcal{Z})$.

Proof. This is a straightforward synthesis of the proof of the A_∞ relation for polygon counting (Proposition 3.11) with the proof of the A_∞ relation for \widehat{CFA} [LOT08, Proposition 7.12]. \square

Remark 4.22. Proposition 4.21 has an interpretation in terms of Fukaya categories. As in Remark 3.16, let TFuk denote the full subcategory of the Fukaya category of $\text{Sym}^g(\Sigma)$ generated by Heegaard tori. Then Proposition 4.21 can be interpreted as saying that $\widehat{CFA}(\alpha, \cdot)$ is an A_∞ -bimodule over TFuk and $\mathcal{A}(\mathcal{Z})$. (Convention 3.4 means that $\widehat{CFA}(\alpha, \cdot)$ is a left-right bimodule; with the usual composition conventions for Heegaard Floer homology $\widehat{CFA}(\alpha, \cdot)$ would be a right-right bimodule.)

There is a type D analogue of the above construction (compare [LOT08, Chapter 6]). Recall that for type D structures, one considers a different orientation convention: in that case, one considers a collection α of bordered attaching curves in Σ which are compatible with $-\mathcal{Z}$.

Now, if $(\Sigma, \alpha, \{\beta^i\}_{i=1}^n, z)$ is an admissible bordered multi-diagram, polygon counts give maps

$$\delta_n^1: \widehat{CF}(\beta^{n-1}, \beta^n, z) \otimes \dots \otimes \widehat{CF}(\beta^1, \beta^2, z) \otimes \widehat{CFD}(\alpha, \beta^1) \rightarrow \mathcal{A}(\mathcal{Z}) \otimes \widehat{CFD}(\alpha, \beta^n),$$

defined as follows:

$$\delta_n^1(\eta^{n-1}, \dots, \eta^1, \mathbf{x}) = \sum_{\substack{\mathbf{y} \in \mathfrak{S}(\boldsymbol{\alpha}, \boldsymbol{\beta}^{i_n}) \\ \vec{\rho} \\ B \in \pi_2(\mathbf{y}, \eta^{n-1}, \dots, \eta^1, \mathbf{x}; \vec{\rho}) \\ \text{ind}(B, \vec{\rho}) = 2-n}} \# \mathcal{M}^B(\mathbf{y}, \eta^{n-1}, \dots, \eta^1, \mathbf{x}; \vec{\rho}) a(-\vec{\rho}) \otimes \mathbf{y},$$

where $\vec{\rho}$ runs over all sequences $\vec{\rho} = (\{\rho_1\}, \dots, \{\rho_k\})$ of (singleton sets of) Reeb chords for which $(\mathbf{x}, \{-\rho_1\}, \dots, \{-\rho_k\})$ is strongly boundary monotone; and if $\vec{\rho} = (\rho_1, \dots, \rho_k)$, then

$$a(-\vec{\rho}) = \prod_{i=1}^k a(-\rho_i).$$

So, for example, δ_1^1 is the differential on $\widehat{CFD}(\boldsymbol{\alpha}, \boldsymbol{\beta}^1)$.

The map δ_n^1 can be naturally extended to a map

$$\delta_n^1: \widehat{CF}(\boldsymbol{\beta}^{n-1}, \boldsymbol{\beta}^n, z) \otimes \dots \otimes \widehat{CF}(\boldsymbol{\beta}^1, \boldsymbol{\beta}^2, z) \otimes \mathcal{A}(\mathcal{Z}) \otimes \widehat{CFD}(\boldsymbol{\alpha}, \boldsymbol{\beta}^1) \rightarrow \mathcal{A}(\mathcal{Z}) \otimes \widehat{CFD}(\boldsymbol{\alpha}, \boldsymbol{\beta}^n)$$

by the formula

$$\widetilde{\delta}_n^1(\eta^{n-1}, \dots, \eta^1, a, \mathbf{x}) = (\mu_2(a, \cdot) \otimes \mathbb{I}_{\widehat{CFD}}) \circ \delta_n^1(\eta^{n-1}, \dots, \eta^1, \mathbf{x}).$$

Proposition 4.23. *The polygon counts defined above satisfy the following A_∞ relation:*

$$\begin{aligned} & \sum_{1 \leq p \leq n} \widetilde{\delta}_{n-p+1}^1(\eta^{i_{n-1} < i_n}, \dots, \eta^{i_p < i_{p+1}}, \delta_{p+1}^1(\eta^{i_{p-1} < i_p}, \dots, \eta^{i_0 < i_1}, \mathbf{x})) \\ & + (\mu_1 \otimes \mathbb{I}) \circ \delta_{n+1}^1(\eta^{i_{n-1} < i_n}, \dots, \eta^{i_0 < i_1}, \mathbf{x}) \\ & + \sum_{1 \leq p \leq q \leq n} \delta_{n-q+p+1}^1(\eta^{i_{n-1} < i_n}, \dots, m_{q-p+1}(\eta^{i_{q-1} < i_q}, \dots, \eta^{i_{p-1} < i_p}), \eta^{i_{p-2} < i_{p-1}}, \dots, \eta^{i_0 < i_1}, \mathbf{x}) \\ & = 0. \end{aligned}$$

Proof. As in the proof of Proposition 4.23, the proof is a combination of the proof of the usual polygon counting A_∞ relation (Proposition 3.11) with the proof of the corresponding relation in the bordered case [LOT08, Proposition 6.7]. \square

Remark 4.24. Continuing with the notation from Remark 4.22, we can think of Proposition 4.23 as giving $\widehat{CFD}(\boldsymbol{\alpha}, \cdot)$ the structure of a type DA bimodule over $\mathcal{A}(-\mathcal{Z})$ and TFuk .

Remark 4.25. Proposition 4.23 can alternatively be thought of as a formal consequence of Proposition 4.21. Before describing this, we give an analogous way of deducing the type D structure relation for $\widehat{CFD}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ from the type A structure relation for $\widehat{CFA}(\boldsymbol{\alpha}, \boldsymbol{\beta})$. Consider the type DD bimodule $X = \widehat{CFDD}(\mathbb{I})$ associated to the identity cobordism from $F(\mathcal{Z})$ to itself. This bimodule was computed in [LOT14b, Definition 1.2]: it is generated by pairs of complementary idempotents, and the differential is given by

$$\partial(I \otimes I') = \sum_{\text{chords } \xi} I a(\xi) \otimes a'(\xi) I'.$$

It follows that $\widehat{CFD}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = X \boxtimes \widehat{CFA}(\boldsymbol{\alpha}, \boldsymbol{\beta})$. (Here, equality denotes a canonical identification, not merely a homotopy equivalence.) Taking this as a definition of $\widehat{CFD}(\boldsymbol{\alpha}, \boldsymbol{\beta})$, the type D structure equation on $\widehat{CFD}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is a formal consequence of the type DD structure

equation on X (as verified directly in [LOT14b, Proposition 3.4]), the type A structure relation on $\widehat{CFA}(\alpha, \beta)$, and the fact that a type DD bimodule tensored with a type A module gives a type D module [LOT15, Section 2.3.2].

In an analogous manner, we could have defined the $\mathcal{A}(-\mathcal{Z})$ -TFuk bimodule structure on $\widehat{CFD}(\alpha, \cdot)$ as $X \boxtimes \widehat{CFA}(\alpha, \cdot)$. Now the desired DA structure equations follow from the A_∞ -bimodule relations (Proposition 4.21), the DD structure relations on X , and the fact that a type DD bimodule tensored with an A_∞ -bimodule gives a type DA structure (see again [LOT15, Section 2.3.2]).

4.4. Complexes of attaching circles and filtered bordered modules. In Proposition 3.19, we described how an \mathbb{I} -filtered chain complex of attaching circles along with another set of attaching circles gives rise to an \mathbb{I} -filtered chain complex.

Our aim here is to prove the analogue in the bordered setting. Specifically, we will show how a chain complex of attaching circles in a Heegaard surface with boundary, along with a further set of bordered attaching curves compatible with a pointed matched circle \mathcal{Z} gives rise to an \mathbb{I} -filtered A_∞ -module.

Definition 4.26. Let Σ be a surface with boundary, z a basepoint in $\partial\Sigma$, and α a complete set of bordered attaching curves compatible with some fixed pointed matched circle \mathcal{Z} . Let $(\{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i < i'}\}_{i, i' \in \mathbb{I}})$ be a chain complex of attaching circles. Suppose moreover that the multi-diagram $(\Sigma, \alpha, \{\beta^i\}_{i \in \mathbb{I}}, z)$ is provincially admissible (see Definition 4.1).

Define the \mathbb{I} -filtered A_∞ -module $\widehat{\mathbf{CFA}}(\alpha, \{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i < i'}\}_{i, i' \in \mathbb{I}}, z)$ over $\mathcal{A}(\mathcal{Z})$ to be

$$\{\widehat{CFA}(\alpha, \beta^i)\}_{i \in \mathbb{I}}$$

together with the A_∞ -homomorphisms

$$F^{i < i'} : \widehat{CFA}(\alpha, \beta^i) \rightarrow \widehat{CFA}(\alpha, \beta^{i'})$$

for $i, i' \in \mathbb{I}$ with $i < i'$ defined by

$$F^{i < i'}(\mathbf{x}, a_1, \dots, a_k) = \sum_{i=i_0 < \dots < i_n=i'} n m_k(\eta^{i_{n-1} < i_n}, \dots, \eta^{i_0 < i_1}, \mathbf{x}, a_1, \dots, a_k).$$

When they are clear from the context, we will drop the indexing set from the notation, writing

$$\widehat{\mathbf{CFA}}(\alpha, \{\beta^i\}, \{\eta^{i < i'}\}) := \widehat{\mathbf{CFA}}(\alpha, \{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i < i'}\}_{i, i' \in \mathbb{I}}).$$

The following is a precise version of the type A case of Theorem 3:

Proposition 4.27. If the diagram $(\Sigma, \alpha, \{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i < i'}\}_{i, i' \in \mathbb{I}}, z)$ is provincially admissible then the object $\widehat{\mathbf{CFA}}(\alpha, \{\beta^i\}, \{\eta^{i < i'}\})$ of Definition 4.26 is an \mathbb{I} -filtered A_∞ -module over $\mathcal{A}(\mathcal{Z})$. Its associated graded object is $\bigoplus_{i \in \mathbb{I}} \widehat{CFA}(\alpha, \beta^i)$. If $(\alpha, \{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i < i'}\}_{i, i' \in \mathbb{I}}, z)$ is admissible (Definition 4.2) then $\widehat{\mathbf{CFA}}(\alpha, \{\beta^i\}, \{\eta^{i < i'}\})$ is bounded.

Proof. The compatibility condition (from Equation (2.3)) is a consequence of the A_∞ relation (Proposition 4.21), together with the compatibility conditions for a chain complex of attaching circles (Equation (3.15)). See also the proof of Proposition 3.19. \square

Remark 4.28. Recall (Remark 3.16) that a chain complex of attaching circles can be viewed as an \mathbb{I} -filtered type D structure (twisted complex) over TFuk. Now, $\widehat{\mathbf{CFA}}$ can be thought of as the tensor product of this filtered type D structure with $\widehat{CFA}(\alpha, \cdot)$, thought of as

a bimodule over TFuk and $\mathcal{A}(\mathcal{Z})$ as in Remark 4.22. Accordingly, this tensor product is naturally an \mathbb{I} -filtered A_∞ -module over $\mathcal{A}(\mathcal{Z})$ (since a type D structure tensored with an A_∞ -bimodule is a type A module [LOT15, Section 2.3.2]).

Similarly, we can form

$$\widehat{\mathrm{CFD}}(\alpha, \{\beta^i\}, \{\eta^{i < i'}\}, z) = \widehat{\mathrm{CFD}}(\alpha, \{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i < i'}\}_{i, i' \in \mathbb{I}}, z),$$

which is an \mathbb{I} -filtered type D structure over $\mathcal{A}(-\mathcal{Z})$. In this case,

$$^{i < i'}\delta^1: \widehat{\mathrm{CFD}}(\alpha, \beta^i) \rightarrow \mathcal{A}(\mathcal{Z}) \otimes \widehat{\mathrm{CFD}}(\alpha, \beta^{i'})$$

is defined by

$$^{i < i'}\delta^1 = \sum_{i=i_1 < \dots < i_n=i'} \delta_n^1(\eta^{i_{n-1} < i_n}, \dots, \eta^{i_1 < i_2}, \mathbf{x}).$$

Here is the more precise version of the type D case of Theorem 3:

Proposition 4.29. *If the diagram $(\Sigma, \alpha, \{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i < i'}\}_{i, i' \in \mathbb{I}}, z)$ is provincially admissible, the object $\widehat{\mathrm{CFD}}(\alpha, \{\beta^i\}, \{\eta^{i < i'}\}, z)$ defined above is an \mathbb{I} -filtered type D structure over $\mathcal{A}(-\mathcal{Z})$. Its associated graded object is $\bigoplus_{i \in \mathbb{I}} \widehat{\mathrm{CFD}}(\alpha, \beta^i, z)$. If $(\alpha, \{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i < i'}\}_{i, i' \in \mathbb{I}}, z)$ is admissible (Definition 4.2) then $\widehat{\mathrm{CFD}}(\alpha, \{\beta^i\}, \{\eta^{i < i'}\}, z)$ is bounded (Definition 2.8).*

Proof. This follows from Proposition 4.23 together with Equation (3.15). \square

4.5. Bordered multi-diagrams with two boundary components and filtered bimodules. We now turn to the generalization to bordered Heegaard diagrams with 2 boundary components.

Definition 4.30. *Let Σ be a compact, oriented surface with two boundary components, $\partial_L \Sigma$ and $\partial_R \Sigma$. Fix pointed matched circles \mathcal{Z}_L and \mathcal{Z}_R consisting of $2k_L$ and $2k_R$ pairs of points $\mathbf{a}_L \subset \partial_L \Sigma$ and $\mathbf{a}_R \subset \partial_R \Sigma$, respectively. A complete set of bordered attaching curves compatible with \mathcal{Z}_L and \mathcal{Z}_R is a collection $\alpha = \{\alpha_1, \dots, \alpha_{g+k_L+k_R}\}$ of curves in Σ such that:*

- *The curves $\alpha_i \in \alpha$ are pairwise disjoint.*
- *$\alpha \cap \partial \Sigma = \partial \alpha = \mathcal{Z}_L \amalg \mathcal{Z}_R$. In particular, each α -arc has both of its endpoints on the same boundary component of Σ . We sometimes abbreviate this condition as $\partial(\Sigma, \alpha) = \mathcal{Z}_L \amalg \mathcal{Z}_R$.*
- *The relative cycles $\{[\alpha_i]\}_{i=1}^{g+k_L+k_R}$, where $[\alpha_i] \in H_1(\Sigma, \partial \Sigma)$, are linearly independent.*

Definition 4.31. *Let α be a complete set of bordered attaching curves in Σ in the sense of Definition 4.30 (compatible with some \mathcal{Z}_L and \mathcal{Z}_R), and let $\{\beta^i\}_{i=1}^n$ be an n -tuple of complete sets of attaching circles (in the sense of Definition 3.1). Fix also an arc $\mathbf{z} \subset \Sigma \setminus (\alpha \cup \bigcup_i \beta^i)$ connecting $\partial_L \Sigma$ and $\partial_R \Sigma$. (Existence of such an arc is not guaranteed by the other hypotheses.) We call the data $(\Sigma, \alpha, \beta^1, \dots, \beta^n, \mathbf{z})$ a bordered multi-diagram with 2 boundary components.*

Define multi-periodic domains in the 2 boundary component case exactly as in Definition 4.2 (with the requirement that the region containing \mathbf{z} has coefficient 0). A multi-periodic domain is called provincial if all of its multiplicities near $\partial \Sigma$ vanish. Admissibility and provincial admissibility are defined as before; these notions can be refined as follows. A domain is called left-provincial (respectively right-provincial) if all its local multiplicities around $\partial_L \Sigma$ (respectively $\partial_R \Sigma$) vanish. The diagram is called left-admissible (respectively

right-admissible) if all its non-zero right-provincial (respectively left-provincial) period domains have both positive and negative local multiplicities.

We can now define bimodule analogues of $\widehat{\mathbf{CFA}}$. This is a straightforward synthesis of the definition of $\widehat{\mathbf{CFAA}}$ from [LOT15, Definition 6.1] and $\widehat{\mathbf{CFA}}$ from Definition 4.26. We sketch this construction.

Fix a chain complex of attaching circles in Σ , $(\mathbb{I}, \{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i_1 < i_2}\}_{i_1, i_2 \in \mathbb{I}})$ and a complete set of bordered attaching curves α (compatible with some \mathcal{Z}_L and \mathcal{Z}_R). Assume that $(\Sigma, \alpha, \{\beta^i\}, \mathbf{z})$ is provincially admissible.

As in [LOT15, Definition 5.5], we can form the *drilled Heegaard surface* Σ_{dr} by attaching a one-handle to Σ to connect the two endpoints of the arc \mathbf{z} . We fix a basepoint $z \in \partial\Sigma_{\text{dr}}$ somewhere on the boundary of the attached one-handle. We can consider $\widehat{\mathbf{CFA}}(\Sigma_{\text{dr}}, \alpha, \{\beta^i\}, \{\eta^{i_1 < i_2}\})$, which is a filtered module over $\mathcal{A}(\mathcal{Z})$, where $\mathcal{Z} = \mathcal{Z}_L \# \mathcal{Z}_R = \partial(\Sigma_{\text{dr}} \cap \alpha)$. Then $\mathcal{A}(\mathcal{Z}_L)$ and $\mathcal{A}(\mathcal{Z}_R)$ are commuting subalgebras of $\mathcal{A}(\mathcal{Z})$.

Via restriction of scalars, we can view $\widehat{\mathbf{CFA}}(\Sigma_{\text{dr}}, \alpha, \{\beta^i\}, \{\eta^{i_1 < i_2}\})$ as a right filtered A_∞ -module over $\mathcal{A}(\mathcal{Z}_L) \otimes \mathcal{A}(\mathcal{Z}_R)$. The category of right (filtered) A_∞ -modules over $\mathcal{A}(\mathcal{Z}_L) \otimes \mathcal{A}(\mathcal{Z}_R)$ is equivalent to the category of right-right (filtered) A_∞ -bimodules over $\mathcal{A}(\mathcal{Z}_L)$ - $\mathcal{A}(\mathcal{Z}_R)$ (see, e.g., [LOT15, Section 2.4.3]); let

$$\widehat{\mathbf{CFAA}}(\Sigma, \alpha, \{\beta^i\}, \{\eta^{i_1 < i_2}\}) = \widehat{\mathbf{CFAA}}(\Sigma, \alpha, \{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i_1 < i_2}\}_{i_1 < i_2 \in \mathbb{I}})$$

denote $\widehat{\mathbf{CFA}}(\Sigma_{\text{dr}}, \alpha, \{\beta^i\}, \{\eta^{i_1 < i_2}\})$ viewed as an $\mathcal{A}(\mathcal{Z}_L)$ - $\mathcal{A}(\mathcal{Z}_R)$ -bimodule. Explicitly, the generators of $\widehat{\mathbf{CFAA}}$ are the generators of $\widehat{\mathbf{CFA}}$. The action of the sequences (a_1, \dots, a_m) in $\mathcal{A}(\mathcal{Z}_L)$ and (b_1, \dots, b_n) in $\mathcal{A}(\mathcal{Z}_R)$ on a generator \mathbf{x} of $\widehat{\mathbf{CFAA}}$ is the sum of the actions on \mathbf{x} of all sequences (c_1, \dots, c_{m+n}) which interleave (a_1, \dots, a_m) and (b_1, \dots, b_n) .

Now, define

$$(4.32) \quad \widehat{\mathbf{CFDA}}(\alpha, \{\beta^i\}, \{\eta^{i_1 < i_2}\}) := \widehat{\mathbf{CFAA}}(\alpha, \{\beta^i\}, \{\eta^{i_1 < i_2}\}) \boxtimes_{\mathcal{A}(\mathcal{Z}_L)} \widehat{\mathbf{CFDD}}(\mathbb{I}_{\mathcal{Z}_L})$$

$$(4.33) \quad \widehat{\mathbf{CFDD}}(\alpha, \{\beta^i\}, \{\eta^{i_1 < i_2}\}) := \widehat{\mathbf{CFAA}}(\alpha, \{\beta^i\}, \{\eta^{i_1 < i_2}\}) \boxtimes_{\mathcal{A}(\mathcal{Z}_L)} \widehat{\mathbf{CFDD}}(\mathbb{I}_{\mathcal{Z}_L}) \\ \boxtimes_{\mathcal{A}(\mathcal{Z}_R)} \widehat{\mathbf{CFDD}}(\mathbb{I}_{\mathcal{Z}_R}).$$

(Compare Remark 4.25.) We spell out this definition more explicitly for the case of $\widehat{\mathbf{CFDA}}$.

As in the connected boundary case, when talking about holomorphic curves we will abuse notation and let Σ denote the result of attaching cylindrical ends to the boundary components of Σ . The definition of an admissible collection of almost-complex structures (Definition 4.4) generalizes in an obvious way to the two boundary component case; fix an admissible collection of almost-complex structures.

Fix also the following data:

- a sequence of sets of Reeb chords $\vec{\rho}^R = (\rho_1^R, \dots, \rho_n^R)$ in \mathcal{Z}_R so that $(\mathbf{x}, \vec{\rho})$ is strongly boundary monotone,
- a collection of generators $\eta^{i_m < i_{m+1}} \in \mathfrak{S}(\beta^{i_m}, \beta^{i_{m+1}})$ for $m = 1, \dots, n-1$.

Define

$$\begin{aligned} & {}_n\delta_k^1(\eta^{i_{n-1} < i_n}, \dots, \eta^{i_1 < i_2}, \mathbf{x}, \boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_k) \\ &= \sum_{\substack{\mathbf{y} \in \mathfrak{S}(\boldsymbol{\alpha}, \boldsymbol{\beta}^{i_n}) \\ \vec{\rho}^L \\ B \in \pi_2(\mathbf{y}, \eta^{i_{n-1}}, \dots, \eta^1, \mathbf{x}; \vec{\rho}^R, \vec{\rho}^L) \\ \text{ind}(B; \vec{\rho}^R, \vec{\rho}^L) = 2-n}} a(-\vec{\rho}^L) \otimes \# \mathcal{M}^B(\mathbf{y}, \eta^{i_{n-1} < i_n}, \dots, \eta^{i_1 < i_2}, \mathbf{x}; \vec{\rho}^L; \vec{\rho}^R) \mathbf{y}, \end{aligned}$$

where here the sum is over all sequences of Reeb chords $\vec{\rho}^L$ in \mathcal{Z}_L , and

$$\# \mathcal{M}^B(\mathbf{y}, \eta^{i_{n-1} < i_n}, \dots, \eta^{i_1 < i_2}, \mathbf{x}; \vec{\rho}^L; \vec{\rho}^R)$$

denotes the counts of holomorphic polygons whose corners (in order) are mapped to \mathbf{y} , $\eta^{i_{n-1} < i_n}, \dots, \eta^{i_1 < i_2}$, and \mathbf{x} ; the Reeb chords appearing along α -arcs which land in $\partial_L \Sigma$ are, in order, $\vec{\rho}^L$; and the sets of Reeb chords appearing along the α -arcs which land in $\partial_R \Sigma$ are, in order, $\vec{\rho}^R$.

Let $\widehat{CFDA}(\boldsymbol{\alpha}, \boldsymbol{\beta}^i)$ denote the vector space generated by $\mathfrak{S}(\boldsymbol{\alpha}, \boldsymbol{\beta}^i)$. We can extend ${}_n\delta_k^1$ linearly to give a map

$$\begin{aligned} & {}_n\delta_k^1: \widehat{CFDA}(\boldsymbol{\beta}^{i_{n-1}}, \boldsymbol{\beta}^{i_n}) \otimes \dots \otimes \widehat{CF}(\boldsymbol{\beta}^{i_1}, \boldsymbol{\beta}^{i_2}, \mathbf{z}) \otimes \widehat{CFDA}(\boldsymbol{\alpha}, \boldsymbol{\beta}^{i_1}) \otimes \overbrace{\mathcal{A}(\mathcal{Z}_R) \otimes \dots \otimes \mathcal{A}(\mathcal{Z}_R)}^k \\ & \rightarrow \mathcal{A}(\mathcal{Z}_L) \otimes \widehat{CFDA}(\boldsymbol{\alpha}, \boldsymbol{\beta}^{i_n}). \end{aligned}$$

Fix a chain complex $(\{\boldsymbol{\beta}^i\}_{i \in \mathbb{I}}, \eta^{i_1 < i_2}, \mathbf{z})$ of attaching circles. Define

$${}^{i < i'}\delta_k^1: \widehat{CFDA}(\boldsymbol{\alpha}, \boldsymbol{\beta}^i) \otimes \overbrace{\mathcal{A}(\mathcal{Z}_R) \otimes \dots \otimes \mathcal{A}(\mathcal{Z}_R)}^k \rightarrow \mathcal{A}(\mathcal{Z}_L) \otimes \widehat{CFDA}(\boldsymbol{\alpha}, \boldsymbol{\beta}^{i'})$$

by

$$(4.34) \quad {}^{i < i'}\delta_k^1(\mathbf{x}, \boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_k) = \sum_n \sum_{i=i_1 < \dots < i_n=i'} {}_n\delta_k^1(\eta^{i_{n-1} < i_n}, \dots, \eta^{i_1 < i_2}, \mathbf{x}, \boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_k).$$

Lemma 4.35. *The two definitions of $\widehat{CFDA}(\boldsymbol{\alpha}, \{\boldsymbol{\beta}^i\}, \{\eta^{i_1 < i_2}\}, \mathbf{z})$, from Formula (4.32) and Formula (4.34), agree.*

Proof. This is immediate from the definitions. (Compare Remark 4.25.) \square

Proposition 4.36. *Let $(\Sigma, \boldsymbol{\alpha}, \{\boldsymbol{\beta}^i\}_{i \in \mathbb{I}}, \mathbf{z})$ be an admissible bordered multi-diagram with 2 boundary components. The above maps ${}^{i < i'}\delta_k^1$ give $\bigoplus_{i \in \mathbb{I}} \widehat{CFDA}(\boldsymbol{\alpha}, \boldsymbol{\beta}^i)$ the structure of an \mathbb{I} -filtered type DA bimodule. Moreover, if the diagram is left-admissible (respectively right-admissible) then the corresponding module is left-bounded (respectively right-bounded).*

Proof. The fact that these maps define a filtered type DA bimodule is a straightforward combination of the compactification of moduli spaces of polygons with the arguments from standard bordered Floer homology, as in [LOT15, Proposition 6.15]. Alternately, this follows from Lemma 4.35. The admissibility and boundedness statements follow exactly along the lines of [LOT15, Lemma 6.17]. \square

5. THE PAIRING THEOREM FOR CHAIN COMPLEXES OF ATTACHING CIRCLES

The first goal of this section is to prove Theorem 4, which we restate here:

Theorem 5. *Let (Σ_1, α^1, z) and (Σ_2, α^2, z) be surfaces-with-boundary, each equipped with complete sets of bordered attaching curves α^1 and α^2 and basepoints $z \in \partial\Sigma_i$ with $\partial(\Sigma_1, \alpha^1) = \mathcal{Z}$ and $\partial(\Sigma_2, \alpha^2) = -\mathcal{Z}$. Let*

$$(\Sigma_1, \mathbb{I}, \{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i < i'}\}_{i, i' \in \mathbb{I}}) \text{ and } (\Sigma_2, \mathbb{J}, \{\gamma^j\}_{j \in \mathbb{J}}, \{\zeta^{j < j'}\}_{j, j' \in \mathbb{J}})$$

be chain complexes of attaching circles, in Σ_1 and Σ_2 respectively. Suppose moreover that both $(\Sigma_1, \alpha, \{\beta^i\}_{i \in \mathbb{I}}, z)$ and $(\Sigma_2, \alpha, \{\gamma^j\}_{j \in \mathbb{J}}, z)$ are provincially admissible, and at least one is admissible. Then, there is an $\mathbb{I} \times \mathbb{J}$ -filtered quasi-isomorphism

$$(5.1) \quad \widehat{\text{CFA}}(\Sigma_1, \alpha^1, \{\beta^i\}_{i \in \mathbb{I}}, z) \boxtimes \widehat{\text{CFD}}(\Sigma_2, \alpha^2, \{\gamma^j\}_{j \in \mathbb{J}}, z) \\ \simeq \widehat{\text{CF}}(\Sigma_1 \cup \Sigma_2, \alpha^1 \cup \alpha^2, (\{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i < i'}\}_{i, i' \in \mathbb{I}}) \# (\{\gamma^j\}_{j \in \mathbb{J}}, \{\zeta^{j < j'}\}_{j, j' \in \mathbb{J}}), z).$$

On the left side, we are taking chain complexes of type *A* structures and type *D* structures associated to chain complexes of attaching circles as explained in Subsection 4.4 and then forming their tensor product, which is an $\mathbb{I} \times \mathbb{J}$ -filtered complex (see Lemma 2.9). On the right side, we are taking the connected sum of the complexes of attaching circles, in the sense of Definition 3.40, and forming the associated chain complex, as in Definition 3.18. In particular, this involves using close approximations of the attaching circles, as in Proposition 3.52.

The proof of Theorem 5 occupies Sections 5.1–5.6. In Section 5.7 we formulate a version of Theorem 5 with weaker admissibility hypotheses. We formulate and sketch a proof of the bimodule analogue of Theorem 5 in Section 5.8. There is a simpler pairing theorem in the case of pairing holomorphic triangles with holomorphic bigons, as described in Section 5.9, which we use in Section 5.10 to show that the surgery exact triangle implied by bordered Floer theory [LOT08] agrees with the original Heegaard Floer surgery exact triangle [OSz04a].

The proof of Theorem 5 has the following steps, as illustrated in Figure 10:

- (1) Form the connected sum of Σ_1 and Σ_2 , and take a limit of almost-complex structures so that holomorphic polygons in $\Sigma_1 \# \Sigma_2$ correspond to pairs of polygons in Σ_1 and Σ_2 which satisfy a matching condition for both the conformal structure of the polygon and the positions of the Reeb chords (Proposition 5.13). The moduli space of such matched polygons is denoted $\mathcal{MM}_{[0]}$, and the complex which counts them is denoted $\mathcal{C}_{[0]}$.
- (2) Consider a new chain complex, $\mathcal{C}_{[t]}$, whose differential counts points in the moduli space $\mathcal{MM}_{[t]}$ of pairs of polygons where the matching condition for the Reeb chords is translated by a real parameter t . The filtered quasi-isomorphism type of this complex is independent of the parameter t (Lemma 5.22). (This is similar to a step in the proof of [LOT15, Theorem 7].)
- (3) Send the t parameter to ∞ (Lemma 5.25). The chain complex $\mathcal{C}_{[t]}$ for large enough t is now identified with a different chain complex, $\mathcal{C}_{[\infty]}$, whose differential counts points in the moduli space \mathcal{XM} of *cross-matched polygons*. Cross-matched polygons consists of two-story buildings $u_1 * u_2$ and $v_1 * v_2$ in Σ_1 and Σ_2 respectively, where the modulus of u_i (in $\text{Conf}(D_n)$) is matched with the modulus of v_i , for $i = 1, 2$; u_1 and v_2 are provincial; and the (relative) positions of the Reeb chords on u_2 and v_1 are matched.

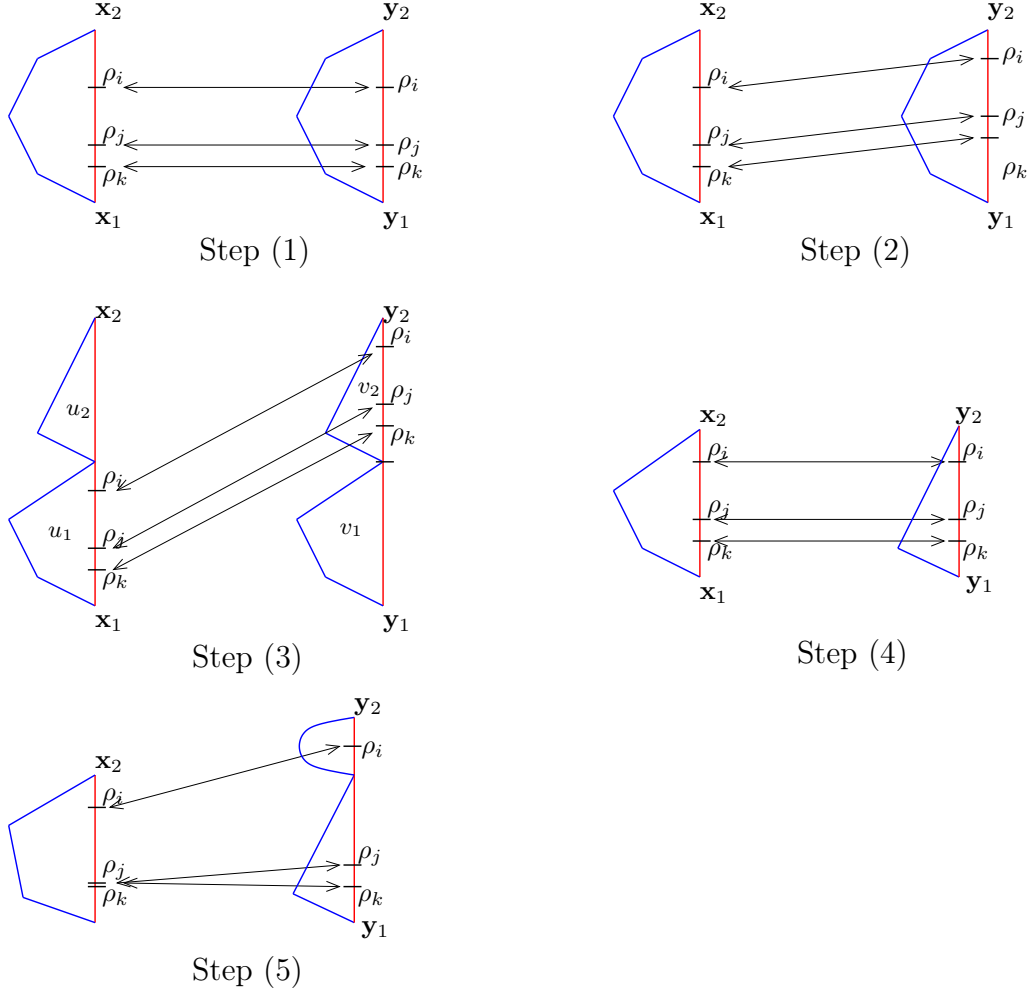


FIGURE 10. **Sketch of the pairing theorem.** Schematic illustration of the curve configurations which contribute to the boundary operators of the chain complexes described in Steps (1)-(5). Note that in the first two steps, the precise positions of the chords are matched (in the second step, up to an overall, pre-specified translation factor); while in the other steps, only the relative positions are matched. Moreover, in the first three steps, there is a matching of the conformal structures of the curves on the two sides, absent from the last two steps.

- (4) Now, move the approximation parameter ϵ in the construction of the attaching circles $\beta^{i \times j}$ and $\gamma^{i \times j}$. If ϵ is sufficiently small, the cross-matched polygons in the previous step can be simplified: the provincial curves v_1 and u_2 are determined uniquely, and hence they can be thrown out (Proposition 5.31). This gives a new chain complex, denoted $\mathcal{C}_{\mathcal{N}}$, counting points in the moduli space \mathcal{N} of *chord-matched polygon pairs*, consisting of a polygon u_1 in Σ_1 and v_2 in Σ_2 , which satisfy a matching condition on the chord heights. At this point, the moduli of the polygons become unconstrained.
- (5) Dilate time in the matching conditions on chord-matched polygon pairs, as in the proof of the usual pairing theorem in bordered Floer homology [LOT08, Chapter 9]. As in that proof, once the dilation parameter is sufficiently large (Proposition 5.34),

the chain complex is identified with a chain complex \mathcal{C}_{tsic} , whose differential counts *trimmed simple ideal-matched polygon pairs* (Definition 5.33).

- (6) Again as in the proof of the original pairing theorem, counts of trimmed simple ideal-matched polygon pairs have an algebraic interpretation, establishing the pairing theorem for polygons.

5.1. Preliminaries. Before launching into the above steps, we make an observation concerning the admissibility hypotheses:

Proposition 5.2. *If both $(\Sigma_1, \alpha^1, \{\beta^i\}_{i \in \mathbb{I}}, z)$ and $(\Sigma_2, \alpha^2, \{\gamma^j\}_{j \in \mathbb{J}}, z)$ are provincially admissible, and one of the two is admissible, then their gluing*

$$(\Sigma_1 \cup \Sigma_2, \alpha^1 \cup \alpha^2, (\{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i < i'}\}_{i, i' \in \mathbb{I}}) \# (\{\gamma^j\}_{j \in \mathbb{J}}, \{\zeta^{j < j'}\}_{j, j' \in \mathbb{J}}), z)$$

is admissible.

Proof. In the usual bordered setting, the analogous result is [LOT08, Lemma 4.33]. The fact that we are dealing here with multi-diagrams causes no additional complications. \square

Thus, the admissibility hypotheses ensure that the right-hand side of Equation (5.1) is well-defined. Moreover, provincial admissibility ensures that the tensor factors appearing on the left-hand side are defined, and the admissibility hypotheses ensure that the tensor product is defined (see Propositions 4.27, 4.29, and Lemma 2.9).

5.2. Holomorphic curves in $\Sigma_1 \# \Sigma_2$ and matched polygons. In this subsection we discuss the limits of polygons in $\Sigma_1 \# \Sigma_2$ when one stretches the neck. The results are analogous to [LOT08, Section 9.1]; we assume the reader is familiar with the treatment there, and highlight the parts where the case of polygons is somewhat more complicated.

Convention 5.3. *Throughout the rest of this section, we fix:*

- Riemann surfaces Σ_1 and Σ_2 .
- A pointed matched circle \mathcal{Z} .
- Complete sets of bordered attaching curves α_1 (respectively α_2) in Σ_1 (respectively Σ_2) compatible with \mathcal{Z} (respectively $-\mathcal{Z}$).
- Complete sets of attaching circles $\beta^1, \dots, \beta^{n_1}$ (respectively $\gamma^1, \dots, \gamma^{n_2}$) in Σ_1 (respectively Σ_2).
- Chains $\eta^{i_1 < i_2} \in \widehat{CF}(\Sigma_1, \beta^{i_1}, \beta^{i_2}, z)$ (respectively $\zeta^{j_1 < j_2} \in \widehat{CF}(\Sigma_2, \gamma^{j_1}, \gamma^{j_2}, z)$) making $(\Sigma_1, \{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i_1 < i_2}\}_{i_1, i_2 \in \mathbb{I}}, z)$ (respectively $(\Sigma_2, \{\gamma^j\}_{j \in \mathbb{J}}, \{\zeta^{j_1 < j_2}\}_{j_1, j_2 \in \mathbb{J}}, z)$) into a chain complex of attaching circles.
- For each ℓ, m , a close approximation $\beta^{i_\ell \times j_m}$ (respectively $\gamma^{i_\ell \times j_m}$) to β^{i_ℓ} (respectively γ^{j_m}).

Recall from Convention 3.32 that ij_k denotes $i_k \times j_k$. Also recall from Definition 3.40 that the chain $\eta^{\dot{i}j_1 < \dot{i}j_2}$ is the nearest-point map applied to $\eta^{i_1 < i_2}$ if $j_1 = j_2$, or is the top generator if $i_1 = i_2$, or else is 0, and similarly for $\zeta^{\dot{i}j_1 < \dot{i}j_2}$ but with the roles of i and j exchanged, and

$$\omega_{\dot{i}j_1 < \dot{i}j_2} = \eta^{\dot{i}j_1 < \dot{i}j_2} \otimes \zeta^{\dot{i}j_1 < \dot{i}j_2}.$$

We find it convenient to make the following notational simplification. Given generators $\mathbf{x}_1 \in \mathfrak{S}(\alpha^1, \beta^i)$ and $\mathbf{x}_2 \in \mathfrak{S}(\alpha^1, \beta^{i'})$, and a sequence $\dot{i}j_1 < \dots < \dot{i}j_n$ with $i = i_1$ and $i' = i_n$, let

$$\pi_2(\mathbf{x}_2, \eta^{\dot{i}j_{n-1} < \dot{i}j_n}, \dots, \eta^{\dot{i}j_1 < \dot{i}j_2}, \mathbf{x}_1)$$

denote the union over all terms \mathbf{w}_m in the chain $\eta^{\tilde{y}_m < \tilde{y}_{m+1}}$, $m = 1, \dots, n-1$, of

$$\pi_2(\mathbf{x}_2, \mathbf{w}_{n-1}, \dots, \mathbf{w}_1, \mathbf{x}_1).$$

We make the corresponding notational simplification on Σ_2 , as well.

Definition 5.4. Let $\mathbf{x} \in \mathfrak{S}(\alpha^1, \beta^{i \times j})$ and $\mathbf{y} \in \mathfrak{S}(\alpha^2, \gamma^{i \times j})$ be a pair of generators with the property that the α^1 -arcs occupied by the generator \mathbf{x} are complementary to the α^2 -arcs occupied by the generator \mathbf{y} , with respect to the identification induced by $\partial(\Sigma_1, \alpha^1) = \mathcal{Z} = -\partial(\Sigma_2, \alpha^2)$. Then we say that \mathbf{x} and \mathbf{y} are a complementary pair of generators. Note that this condition is equivalent to the condition that

$$\mathbf{x} \cup \mathbf{y} \in \mathfrak{S}(\alpha^1 \cup \alpha^2, \beta^{i \times j} \cup \gamma^{i \times j}).$$

We denote this induced generator by $\mathbf{x} \# \mathbf{y} := \mathbf{x} \cup \mathbf{y}$.

Given two complementary pairs of generators $(\mathbf{x}_1, \mathbf{y}_1)$ and $(\mathbf{x}_2, \mathbf{y}_2)$, there is an obvious (injective) map

$$\begin{aligned} \pi_2(\mathbf{x}_2 \# \mathbf{y}_2, \eta^{\tilde{y}_{n-1} < \tilde{y}_n} \# \zeta^{\tilde{y}_{n-1} < \tilde{y}_n}, \dots, \eta^{\tilde{y}_1 < \tilde{y}_2} \# \zeta^{\tilde{y}_1 < \tilde{y}_2}, \mathbf{x}_1 \# \mathbf{y}_1) \\ \rightarrow \pi_2(\mathbf{x}_2, \eta^{\tilde{y}_{n-1} < \tilde{y}_n}, \dots, \eta^{\tilde{y}_1 < \tilde{y}_2}, \mathbf{x}_1) \times \pi_2(\mathbf{y}_2, \zeta^{\tilde{y}_{n-1} < \tilde{y}_n}, \dots, \zeta^{\tilde{y}_1 < \tilde{y}_2}, \mathbf{y}_1). \end{aligned}$$

We call the image of this map the set of matched domains (connecting these generators). If (B_1, B_2) is in the image, then its preimage is denoted $B_1 \natural B_2$.

Our next goal is to define matched holomorphic curves. Before doing so, we need a little more notation. Fix generators \mathbf{x}_1 and \mathbf{y}_1 , a source S_1 , Reeb chords ρ_1, \dots, ρ_m and a homology class B_1 . We will define a map

$$(\text{ev} \tilde{\times} \kappa)_1: \mathcal{M}^{B_1}(\mathbf{y}_1, \eta^{\tilde{y}_{n-1} < \tilde{y}_n}, \dots, \eta^{\tilde{y}_1 < \tilde{y}_2}, \mathbf{x}_1; \{\rho_1\}, \dots, \{\rho_m\}; S_1) \rightarrow (\mathbb{R}^m \times \mathbb{R}^{n-1})/\mathbb{R}$$

(where \mathbb{R} acts on $\mathbb{R}^m \times \mathbb{R}^{n-1} \cong \mathbb{R}^{n+m-1}$ by diagonal translation). To this end, consider an element $(j, u) \in \mathcal{M}^{B_1}(\mathbf{y}_1, \eta^{\tilde{y}_{n-1} < \tilde{y}_n}, \dots, \eta^{\tilde{y}_1 < \tilde{y}_2}, \mathbf{x}_1; \{\rho_1\}, \dots, \{\rho_m\}; S_1)$. There is a corresponding polygon $\kappa(j, u) := j \in \text{Conf}(D_{n+1})$. As we have been suppressing j from the notation, we will often abuse notation and write $\kappa(u)$ instead of $\kappa(j, u)$. Now, consider a biholomorphic map $\kappa(u) \rightarrow [0, 1] \times \mathbb{R}$ that sends the corners of $\kappa(u)$ corresponding to \mathbf{x}_1 and \mathbf{y}_1 to $-\infty$ and ∞ , respectively. (This map is well-defined up to translation.) The remaining corners of $\kappa(u)$ give points $(1, t'_1), \dots, (1, t'_{n-1}) \in \{0\} \times \mathbb{R}$ and the coordinates of the Reeb chords of u give points $(0, t_1), \dots, (0, t_m) \in \{0\} \times \mathbb{R}$. (Here, the orderings are chosen so that $t'_1 < \dots < t'_{n-1}$ and $t_1 < \dots < t_m$.) Define

$$(\text{ev} \tilde{\times} \kappa)_1(u) = (t_1, \dots, t_m, t'_1, \dots, t'_{n-1}) \in (\mathbb{R}^m \times \mathbb{R}^{n-1})/\mathbb{R}.$$

We define a map $(\text{ev} \tilde{\times} \kappa)_2$ on the moduli space of curves in Σ_2 similarly.

Definition 5.5. Fix $i < i'$ and $j < j'$, and let

$$\begin{aligned} \mathbf{x}_1 &\in \mathfrak{S}(\alpha^1, \beta^{i \times j}), & \mathbf{x}_2 &\in \mathfrak{S}(\alpha^1, \beta^{i' \times j'}), \\ \mathbf{y}_1 &\in \mathfrak{S}(\alpha^2, \gamma^{i \times j}), & \mathbf{y}_2 &\in \mathfrak{S}(\alpha^2, \gamma^{i' \times j'}) \end{aligned}$$

be generators with the property that \mathbf{x}_1 and \mathbf{y}_1 is a complementary pair of generators, and \mathbf{x}_2 and \mathbf{y}_2 is a complementary pair of generators. Fix a sequence $i \times j = \tilde{y}_1 < \dots < \tilde{y}_n = i' \times j'$ and matched homology classes

$$\begin{aligned} B_1 &\in \pi_2(\mathbf{x}_2, \eta^{\tilde{y}_{n-1} < \tilde{y}_n}, \dots, \eta^{\tilde{y}_1 < \tilde{y}_2}, \mathbf{x}_1) \\ B_2 &\in \pi_2(\mathbf{y}_2, \zeta^{\tilde{y}_{n-1} < \tilde{y}_n}, \dots, \zeta^{\tilde{y}_1 < \tilde{y}_2}, \mathbf{y}_1). \end{aligned}$$

Fix also punctured Riemann surfaces with boundary S_1 and S_2 and a sequence of Reeb chords (ρ_1, \dots, ρ_m) .

Define the moduli space of matched polygons in the homology classes B_1 and B_2 with sources S_1 and S_2 and Reeb chords (ρ_1, \dots, ρ_m) to be

$$(5.6) \quad \mathcal{MM}_{[0]}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1, \mathbf{x}_2 \# \mathbf{y}_2; \rho_1, \dots, \rho_m; S_1, S_2) \\ := \mathcal{M}^{B_1}(\mathbf{y}_1, \eta^{\dot{j}_{n-1} < \dot{j}_n}, \dots, \eta^{\dot{j}_1 < \dot{j}_2}, \mathbf{x}_1; \{\rho_1\}, \dots, \{\rho_m\}; S_1) \\ \times_{\mathbb{I}} \mathcal{M}^{B_2}(\mathbf{y}_2, \zeta^{\dot{j}_{n-1} < \dot{j}_n}, \dots, \zeta^{\dot{j}_1 < \dot{j}_2}, \mathbf{x}_2; \{-\rho_1\}, \dots, \{-\rho_m\}; S_2),$$

where by $\times_{\mathbb{I}}$ we mean the pairs (u_1, u_2) with $(\text{ev} \tilde{\times} \kappa)_1(u_1) = (\text{ev} \tilde{\times} \kappa)_2(u_2)$.

Note that there is a degenerate case of bigons with no Reeb chords (i.e., $n = 1$ and $m = 0$), and correspondingly there are no evaluation maps.

(The notation $[0]$ comes from the fact that these moduli spaces will fit into a one-parameter family indexed by a real number t , and these correspond to $t = 0$; see Definition 5.14.)

The expected dimension of $\mathcal{MM}_{[0]}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1, \mathbf{x}_2 \# \mathbf{y}_2; \rho_1, \dots, \rho_m; S_1, S_2)$ is given in terms of

$$(5.7) \quad \text{ind}(B_1, S_1; B_2, S_2; \rho_1, \dots, \rho_m) := \text{ind}(B_1, S_1; \{\rho_1\}, \dots, \{\rho_m\}) \\ + \text{ind}(B_2, S_2; \{-\rho_1\}, \dots, \{-\rho_m\}) - m \\ = \left(\frac{3-n}{2} \right) (g_1 + g_2) - \chi(S_1) - \chi(S_2) + 2e(B_1) + 2e(B_2) + m$$

according to the following:

Lemma 5.8. *For a generic, admissible family of almost-complex structures for Σ_1 and Σ_2 , the moduli space $\mathcal{MM}_{[0]}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1, \mathbf{x}_2 \# \mathbf{y}_2; \rho_1, \dots, \rho_m; S_1, S_2)$ is transversally cut out, and so is a (non-compact) manifold whose dimension is $\text{ind}(B_1, S_1; B_2, S_2; \rho_1, \dots, \rho_m) + n - 2$ (Formula (5.7)).*

Proof. This follows from standard transversality results. See [LOT08, Lemma 9.4] for the corresponding statement in the original bordered setting. \square

Of course, we want to count embedded holomorphic curves; and as usual this condition determines the Euler characteristic of S . Given

$$B \in \pi_2(\mathbf{x}_2 \# \mathbf{y}_2, \eta^{\dot{j}_{n-1} < \dot{j}_n} \# \zeta^{\dot{j}_{n-1} < \dot{j}_n}, \dots, \eta^{\dot{j}_1 < \dot{j}_2} \# \zeta^{\dot{j}_1 < \dot{j}_2}, \mathbf{x}_1 \# \mathbf{y}_1),$$

following [Sar11], the embedded Euler characteristic and index in the class B are given by

$$(5.9) \quad \chi_{\text{emb}}(B) := g + e(B) - n_{\mathbf{x}_0}(B) - n_{\mathbf{x}_n}(B) - \sum_{n \geq j > \ell \geq 1} \partial_j(B) \cdot \partial_\ell(B)$$

$$(5.10) \quad \text{ind}_{\text{emb}}(B) := e(B) + n_{\mathbf{x}_0}(B) + n_{\mathbf{x}_n}(B) - \left(\frac{n-1}{2} \right) g + \sum_{n \geq j > \ell \geq 1} \partial_j(B) \cdot \partial_\ell(B).$$

(Compare Proposition 4.9.)

Definition 5.11. *The moduli space $\mathcal{MM}_{[0]}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1, \mathbf{x}_2 \# \mathbf{y}_2)$ of embedded matched polygons is the union of the spaces $\mathcal{MM}_{[0]}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1, \mathbf{x}_2 \# \mathbf{y}_2; \rho_1, \dots, \rho_m; S_1, S_2)$ over all*

sequences of Reeb chords and all sources S_1, S_2 with

$$\chi(S_1 \natural S_2) = \chi_{\text{emb}}(B_1 \natural B_2).$$

(Here, $S_1 \natural S_2$ denotes the gluing of S_1 and S_2 at the corresponding punctures. Note that the Euler characteristic of the glued surface $S_1 \natural S_2$ is $\chi(S_1 \natural S_2) = \chi(S_1) + \chi(S_2) - m$.)

Lemma 5.12. *A matched pair of polygons (u_1, u_2) is in the corresponding embedded matched moduli space if and only if both u_i are embedded. Moreover, the expected dimension of $\mathcal{MM}_{[0]}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1, \mathbf{x}_2 \# \mathbf{y}_2)$ is given by $\text{ind}_{\text{emb}}(B_1 \natural B_2) + n - 2$.*

(Compare [LOT08, Lemma 9.8].)

Proof. To see that the u_i are embedded, it suffices to show that $\chi(S_i)$ is given by Formula (4.10). We have

$$\begin{aligned} \chi(S_1 \natural S_2) &= \chi(S_1) + \chi(S_2) - m \\ &\geq \chi_{\text{emb}}(u_1) + \chi_{\text{emb}}(u_2) - m \\ &= g_1 + e(B_1) - n_{\mathbf{x}_0}(B_1) - n_{\mathbf{x}_n}(B_1) - \sum_{n \geq j > \ell \geq 1} \partial_j(B_1) \cdot \partial_\ell(B_1) \\ &\quad - \sum_i \iota(\{\rho_i\}) - \sum_{i < j} L(\rho_i, \rho_j) \\ &\quad + g_2 + e(B_2) - n_{\mathbf{x}_0}(B_2) - n_{\mathbf{x}_n}(B_2) - \sum_{n \geq j > \ell \geq 1} \partial_j(B_2) \cdot \partial_\ell(B_2) \\ &\quad - \sum_i \iota(\{-\rho_i\}) - \sum_{i < j} L(-\rho_i, -\rho_j) - m \\ &= \chi_{\text{emb}}(B_1 \natural B_2), \end{aligned}$$

where the last equality uses the facts that

$$\begin{aligned} \sum_i \iota(\{\rho_i\}) + \sum_i \iota(\{-\rho_i\}) &= -m \\ \sum_{i < j} L(\rho_i, \rho_j) + \sum_{i < j} L(-\rho_i, -\rho_j) &= 0. \end{aligned}$$

Thus, if $\chi(S_1 \natural S_2) = \chi_{\text{emb}}(B_1 \natural B_2)$ we must have $\chi(S_i) = \chi_{\text{emb}}(u_i)$, as desired.

The fact that the expected dimension is given by $\text{ind}_{\text{emb}}(B_1 \natural B_2) + n - 2$ follows from Formula (5.10) and Lemma 5.8. \square

The moduli spaces defined above can be used to construct an $\mathbb{I} \times \mathbb{J}$ -filtered chain complex $\mathcal{C}_{[0]} = \{\mathcal{C}_{[0]}^{i \times j}, D^{i \times j < i' \times j'}\}$ defined by $\mathcal{C}_{[0]}^{i \times j} = \overline{CF}(\alpha^1 \cup \alpha^2, \beta^{i \times j} \cup \gamma^{i \times j}, z)$ and

$$\begin{aligned} D^{i \times j < i' \times j'}(\mathbf{x}_1 \# \mathbf{y}_1) &= \sum_{\substack{\mathbf{x}_2 \times \mathbf{y}_2 \in \mathfrak{S}(\alpha^1 \cup \alpha^2, \beta^{i' \times j'} \cup \gamma^{i' \times j'}) \\ i_1 < i_2 < \dots < i_n \\ (B_1, B_2) \text{ s.t. } \text{ind}_{\text{emb}}(B_1 \natural B_2) = 2 - n}} \# \mathcal{MM}_{[0]}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1, \mathbf{x}_2 \# \mathbf{y}_2) \mathbf{x}_2 \# \mathbf{y}_2. \end{aligned}$$

The following is a generalization of Lemma 3.45, as well as [LOT08, Theorem 9.10]:

Proposition 5.13. *For sufficiently long connect sum parameters on $\Sigma_1 \# \Sigma_2$, there is an identification of chain complexes*

$$\mathcal{C}_{[0]} \cong \widehat{\mathbf{CF}}(\Sigma_1 \cup \Sigma_2, \alpha^1 \cup \alpha^2, (\{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i < i'}\}_{i, i' \in \mathbb{I}}) \# (\{\gamma^j\}_{j \in \mathbb{J}}, \{\zeta^{j < j'}\}_{j, j' \in \mathbb{J}}), z).$$

Proof. This is a standard neck stretching argument; see [LOT08, Theorem 9.10] for the result for the case of bigons. \square

5.3. Time translation. We now introduce a time translation parameter $t \in \mathbb{R}$.

$$\tau_t: (\mathbb{R}^m \times \mathbb{R}^{n-1})/\mathbb{R} \rightarrow (\mathbb{R}^m \times \mathbb{R}^{n-1})/\mathbb{R}$$

be the map

$$\tau_t(t_1, \dots, t_m, t'_1, \dots, t'_{n-1}) = (t + t_1, \dots, t + t_m, t'_1, \dots, t'_{n-1}).$$

Definition 5.14. *The moduli space of t -slid matched polygons is defined exactly like the moduli space of matched polygons (Definition 5.5) except with Equation 5.6 replaced by*

$$(5.15) \quad \begin{aligned} \mathcal{MM}_{[t]}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1, \mathbf{x}_2 \# \mathbf{y}_2; \rho_1, \dots, \rho_m; S_1, S_2) \\ := \mathcal{M}^{B_1}(\mathbf{x}_2, \eta^{\dot{y}_{n-1} < \dot{y}_n}, \dots, \eta^{\dot{y}_1 < \dot{y}_2}, \mathbf{x}_1; \{\rho_1\}, \dots, \{\rho_m\}; S_1) \\ \times_{\tau_t} \mathcal{M}^{B_2}(\mathbf{y}_2, \zeta^{\dot{y}_{n-1} < \dot{y}_n}, \dots, \zeta^{\dot{y}_1 < \dot{y}_2}, \mathbf{y}_1; \{-\rho_1\}, \dots, \{-\rho_m\}; S_2), \end{aligned}$$

where by \times_{τ_t} we mean the pairs (u_1, u_2) with $\tau_t((\text{ev} \times \kappa)_1(u_1)) = (\text{ev} \times \kappa)_2(u_2)$.

The moduli space of embedded t -slid matched polygons $\mathcal{MM}_{[t]}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1, \mathbf{x}_2 \# \mathbf{y}_2)$ is the union of the spaces $\mathcal{MM}_{[t]}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1; \mathbf{x}_2 \# \mathbf{y}_2; \rho_1, \dots, \rho_m; S_1, S_2)$ over all sequences of Reeb chords and all sources S_1, S_2 with $\chi(S_1 \natural S_2) = \chi_{\text{emb}}(B_1 \natural B_2)$.

The notion of a 0-slid matched polygon, of course, coincides with the notion of a matched polygon as in Definition 5.5. Moreover, the moduli space of t -slid matched bigons ($n = 1$) is independent of t , as is the degenerate case of t -slid matched polygons with no Reeb chords ($m = 0$).

The obvious analogue of Lemma 5.8 holds for the t -slid matched moduli spaces:

Lemma 5.16. *For any fixed t and generic J (depending on t), the moduli spaces of t -slid matched polygons $\mathcal{MM}_{[t]}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1; \mathbf{x}_2 \# \mathbf{y}_2; \rho_1, \dots, \rho_m; S_1, S_2)$ are transversally cut out. Similarly, for fixed, generic J and generic t (depending on J), the moduli spaces*

$$\mathcal{MM}_{[t]}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1; \mathbf{x}_2 \# \mathbf{y}_2; \rho_1, \dots, \rho_m; S_1, S_2)$$

are transversally cut out. The dimension of $\mathcal{MM}_{[t]}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1; \mathbf{x}_2 \# \mathbf{y}_2; \rho_1, \dots, \rho_m; S_1, S_2)$ is given by $\text{ind}(B_1, S_1; B_2, S_2; \rho_1, \dots, \rho_m) + n - 2$.

Proof. Like Lemma 5.8, this follows from standard arguments (which we do not spell out here). \square

Lemma 5.12 still applies to show that the embedded moduli spaces consist of embedded curves.

The moduli spaces of t -slid matched polygons can be used to define a chain complex $\mathcal{C}_{[t]}$ generalizing $\mathcal{C}_{[0]}$. Define $\mathcal{C}_{[t]} = \{\mathcal{C}_{[t]}^{i \times j}, D_{[t]}^{i \times j < i' \times j'}\}$ to be $\mathcal{C}_{[t]}^{i \times j} = \widehat{\mathbf{CF}}(\alpha^1 \cup \alpha^2, \beta^{i \times j} \cup \gamma^{i \times j}, z)$

with differential

$$D_{[t]}^{i \times j < i' \times j'}(\mathbf{x}_1 \# \mathbf{y}_1) = \sum_{\substack{\mathbf{x}_2 \times \mathbf{y}_2 \in \mathfrak{S}(\alpha^1 \cup \alpha^2, \beta^{i' \times j'} \cup \gamma^{i' \times j'}) \\ ij_1 < ij_2 < \dots < ij_n \\ (B_1, B_2) \text{ s.t. } \text{ind}_{\text{emb}}(B_1 \natural B_2) = 2 - n}} \# \mathcal{MM}_{[t]}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1; \mathbf{x}_2 \# \mathbf{y}_2) \mathbf{x}_2 \# \mathbf{y}_2.$$

This is the same as $\mathcal{C}_{[0]}$, except that the differential counts t -slid matched polygons rather than simply matched polygons.

Remark 5.17. Moduli spaces with a shift in the time parameter appeared in the proof of [LOT15, Theorem 7].

Lemma 5.18. *For a generic J and generic $t \in \mathbb{R}$, $\mathcal{C}_{[t]}$ is an $\mathbb{I} \times \mathbb{J}$ -filtered chain complex.*

Proof. As usual, this follows from looking at ends of one-dimensional moduli spaces; compare [LOT08, Proposition 9.19]. A one-dimensional moduli space of t -slid matched polygons can have the following kinds of ends:

- (e-1) A two-story t -matched polygon (i.e., this looks like two copies of the picture from Figure 4 parts (d) or (e)).
- (e-2) A t -matched polygon, juxtaposed with a pair of polygons, with boundaries on the β -curves (i.e., this looks like two copies of the picture from Figure 4 parts (b) or (c)).
- (e-3) A t -slid matched comb, with a single curve at $e\infty$ which is a join component (in the sense of [LOT08, Section 5.3]).
- (e-4) A t -slid matched comb, with a single curve at $e\infty$ which is a split component (in the sense of [LOT08, Section 5.3]).

Ends of Type (e-2) cancel in pairs (just like in the proof of Proposition 3.19), since the $\{\eta^{i \times j < i' \times j'} \# \zeta^{i \times j < i' \times j'}\}_{i \times j < i' \times j' \in \mathbb{I} \times \mathbb{J}}$ form a chain complex of attaching circles (according to Proposition 3.52).

Ends of Types (e-3) and (e-4) are the only ends involving curves at $e\infty$, as in the proof of [LOT08, Proposition 9.17]. Moreover, these ends cancel in pairs, as in the proof of [LOT08, Proposition 9.18].

The remaining terms, corresponding to ends of Type (e-1), are all counted in the $\mathbf{x}_2 \times \mathbf{y}_2$ coefficient of

$$D_{[t]}^{i_1 \times j_1 < i' \times j'} \circ D_{[t]}^{i \times j < i_1 \times j_1}(\mathbf{x}_1 \times \mathbf{y}_1) + \partial \circ D_{[t]}^{i \times j < i' \times j'}(\mathbf{x}_1 \times \mathbf{y}_1) + D_{[t]}^{i \times j < i' \times j'} \circ \partial(\mathbf{x}_1 \times \mathbf{y}_1).$$

(The last two terms correspond to when one of the degenerating polygons is a bigon.) \square

We turn next to proving independence of the translation parameter t . Again, we need more notation.

Fix a smooth function $\tau_{t_1, t_2}^1 : \mathbb{R} \rightarrow \mathbb{R}$ so that there is a constant N with

$$(5.19) \quad \tau_{t_1, t_2}^1(t) = \begin{cases} t + t_1 & \text{if } t < -N \\ t + t_2 & \text{if } t > N \end{cases}$$

This induces a function

$$\tau_{t_1, t_2} : \mathbb{R}^m \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}^m \times \mathbb{R}^{n-1}$$

by

$$\tau_{t_1, t_2}(x_1, \dots, x_m, y_1, \dots, y_{n-1}) = (\tau_{t_1, t_2}^1(x_1), \dots, \tau_{t_1, t_2}^1(x_m), y_1, \dots, y_{n-1}).$$

Let $\pi: \mathbb{R}^{m+n-1} \rightarrow \mathbb{R}^{m+n-1}/\mathbb{R}$ denote projection. Let

$$\widetilde{\mathcal{M}}^{B_1} = \{(u, p) \in \mathcal{M}^{B_1} \times \mathbb{R}^{m+n-1} \mid (\text{ev} \tilde{\times} \kappa)_1(u) = \pi(p) \in (\mathbb{R}^m \times \mathbb{R}^{n-1})/\mathbb{R}\}.$$

So, $\widetilde{\mathcal{M}}^{B_1} \cong \mathcal{M}^{B_1} \times \mathbb{R}$. Define $\widetilde{\mathcal{M}}^{B_2}$ similarly. There are well-defined maps $(\text{ev} \tilde{\times} \kappa)_i: \widetilde{\mathcal{M}}^{B_i} \rightarrow \mathbb{R}^m \times \mathbb{R}^{n-1}$.

Definition 5.20. *The moduli space of t_1 - t_2 -slid-matched polygons in B_1 and B_2 is defined exactly like the moduli space of matched polygons (Definition 5.5) except with Formula (5.6) replaced by*

$$(5.21) \quad \mathcal{MM}_{[t_1; t_2]}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1, \mathbf{x}_2 \# \mathbf{y}_2; \rho_1, \dots, \rho_m; S_1, S_2) \\ := \widetilde{\mathcal{M}}^{B_1}(\mathbf{x}_2, \eta^{\dot{y}_{n-1} < \dot{y}_n}, \dots, \eta^{\dot{y}_1 < \dot{y}_2}, \mathbf{x}_1; \{\rho_1\}, \dots, \{\rho_m\}; S_1) \\ \times_{\tau_{t_1, t_2}} \widetilde{\mathcal{M}}^{B_2}(\mathbf{y}_2, \zeta^{\dot{y}_{n-1} < \dot{y}_n}, \dots, \zeta^{\dot{y}_1 < \dot{y}_2}, \mathbf{y}_1; \{-\rho_1\}, \dots, \{-\rho_m\}; S_2),$$

where by $\times_{\tau_{t_1, t_2}}$ we mean the pairs (u_1, u_2) with $\tau_{t_1, t_2}((\text{ev} \tilde{\times} \kappa)_1(u_1)) = (\text{ev} \tilde{\times} \kappa)_2(u_2)$.

The moduli space of embedded t_1 - t_2 -slid matched polygons $\mathcal{MM}_{[t_1; t_2]}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1, \mathbf{x}_2 \# \mathbf{y}_2)$ is the union of the spaces $\mathcal{MM}_{[t_1; t_2]}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1; \mathbf{x}_2 \# \mathbf{y}_2; \rho_1, \dots, \rho_m; S_1, S_2)$ over all sequences of Reeb chords and all sources S_1, S_2 with $\chi(S_1 \natural S_2) = \chi_{\text{emb}}(B_1 \natural B_2)$.

Note that the space $\mathcal{MM}_{[t_1; t_2]}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1, \mathbf{x}_2 \# \mathbf{y}_2; \rho_1, \dots, \rho_m; S_1, S_2)$ is one dimension larger than the space $\mathcal{MM}_{[t]}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1, \mathbf{x}_2 \# \mathbf{y}_2; \rho_1, \dots, \rho_m; S_1, S_2)$.

In the case of no Reeb chords ($m = 0$), if $n > 1$ then the moduli space

$$\mathcal{MM}_{[t_1; t_2]}^{B_1 \natural B_2} \cong \mathbb{R} \times \mathcal{M}^{B_1} \times_{\kappa_{B_1} = \kappa_{B_2}} \mathcal{M}^{B_2}.$$

is never rigid. In the case of bigons with no Reeb chords ($m = 0, n = 1$), the moduli space $\mathcal{MM}_{[t_1; t_2]}^{B_1 \natural B_2}$ is rigid only in the special case that B_1 and B_2 are the trivial (all 0) domain of bigons. We do consider these \mathbb{R} -invariant bigons (unions of trivial strips) to be t_1 - t_2 -slid matched polygons.

The obvious analogues of Lemmas 5.12 and 5.16 hold, namely, the curves in the embedded t_1 - t_2 -slid matched moduli spaces are embedded and for generic J and generic t_1, t_2 and τ_{t_1, t_2} these moduli spaces are transversally cut out.

Counting t_1 - t_2 -slid matched polygons furnishes the chain homotopy equivalence used in the following:

Lemma 5.22. *Given generic J and generic $t_1, t_2 \in \mathbb{R}$, there is a filtered chain homotopy equivalence $\mathcal{C}_{[t_1]} \simeq \mathcal{C}_{[t_2]}$.*

Proof. Define a map $f_{[t_1; t_2]}: \mathcal{C}_{[t_1]} \rightarrow \mathcal{C}_{[t_2]}$ by

$$f(\mathbf{x}_1 \# \mathbf{y}_1) = \sum_{\substack{\mathbf{x}_2 \times \mathbf{y}_2 \in \mathfrak{S}(\alpha^1 \cup \alpha^2, \beta^{i'} \times j' \cup \gamma^{i'} \times j') \\ \dot{y}_1 < \dot{y}_2 < \dots < \dot{y}_n \\ (B_1, B_2) \text{ s.t. } \text{ind}_{\text{emb}}(B_1 \natural B_2) = 1 - n}} \# \mathcal{MM}_{[t_1; t_2]}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1; \mathbf{x}_2 \# \mathbf{y}_1) \mathbf{x}_2 \# \mathbf{y}_2.$$

The verification that f is a chain map goes by considering the ends of one-dimensional moduli spaces $\mathcal{MM}_{[t_1; t_2]}^{B_1 \natural C}(\mathbf{x}_1 \# \mathbf{y}_1, \mathbf{x}_2 \# \mathbf{y}_2)$, which are of the following kinds:

- (e-1) Pairs of two-story polygons $u_1 * u_2$ (on Σ_1) and $v_1 * v_2$ (on Σ_2), where (u_1, v_1) is a t_1 - t_2 -slid matched polygon and (u_2, v_2) is a t_2 -slid matched polygon; here, each of the four polygons u_1, u_2, v_1 , and v_2 has an edge mapped into α .

- (e-2) Pairs of two-story polygons $u_1 * u_2$ (on Σ_1) and $v_1 * v_2$ (on Σ_2), where (u_1, v_1) is a t_1 -slid matched polygon and (u_2, v_2) is a t_1 - t_2 -slid matched polygon; here, each of the four polygons u_1 , u_2 , v_1 , and v_2 has an edge mapped into α .
- (e-3) Pairs of two-story polygons $u_1 * u_2$ (on Σ_1) and $v_1 * v_2$ (on Σ_2), where only u_1 and v_1 have edges mapped into α ; in this case, so (u_1, v_1) is a t_1 - t_2 -slid matched polygon and u_2 and v_2 are ordinary, provincial polygons.
- (e-4) Triples (u_1, w, v_2) , where w is a curve at $e\infty$.

The count of ends of Type (e-1) contributes $\partial_{[t_2]} \circ f_{[t_1; t_2]}$, while the count of ends of Type (e-2) contributes $f_{[t_1; t_2]} \circ \partial_{[t_1]}$. Ends of Type (e-3) cancel in pairs, because the $\eta^{i \times j < i' \times j'} \# \zeta^{i \times j < i' \times j'}$ form a chain complex of attaching circles. Ends of Type (e-4) cancel in pairs, corresponding to viewing a split component for u_1 (respectively u_2) as a join component for u_2 (respectively u_1), or vice-versa (as in the proof of Lemma 5.18 or [LOT08, Proposition 9.18]).

The composition $f_{[t_1; t_2]} \circ f_{[t_2; t_1]}$ is chain homotopic to the identity map. The chain homotopy counts polygons in a one-parameter family of moduli spaces of the form $\mathcal{MM}_{[t_1; t_1]}^{B_1 \natural B_2}$ indexed by a real parameter T which varies the choice of interpolating function (Equation (5.19)) implicit in the definition of the moduli space. \square

5.4. Translating time to ∞ and cross-matched polygons. Consider the moduli space $\mathcal{M}^{B_1}(\mathbf{y}_1, \eta^{\tilde{j}_{n-1} < \tilde{j}_n}, \dots, \eta^{\tilde{j}_1 < \tilde{j}_2}, \mathbf{x}_1; \{\rho_1\}, \dots, \{\rho_m\}; S_1)$. The \mathbb{R} -coordinates of the Reeb chords give an evaluation map

$$\text{ev}_{B_1} : \mathcal{M}^{B_1}(\mathbf{y}_1, \eta^{\tilde{j}_{n-1} < \tilde{j}_n}, \dots, \eta^{\tilde{j}_1 < \tilde{j}_2}, \mathbf{x}_1; \{\rho_1\}, \dots, \{\rho_m\}; S_1) \rightarrow \mathbb{R}^m / \mathbb{R}.$$

Similarly, the conformal structure on the source gives a forgetful map

$$\kappa_{B_1} : \mathcal{M}^{B_1}(\mathbf{y}_1, \eta^{\tilde{j}_{n-1} < \tilde{j}_n}, \dots, \eta^{\tilde{j}_1 < \tilde{j}_2}, \mathbf{x}_1; \{\rho_1\}, \dots, \{\rho_m\}; S_1) \rightarrow \text{Conf}(D_{n+1})$$

(compare Equation (3.43)). In other words, these maps can be obtained from $(\text{ev} \tilde{\times} \kappa)_1$ by projecting to $\mathbb{R}^m / \mathbb{R}$ or $\mathbb{R}^{n-1} / \mathbb{R}$, respectively:

$$\begin{array}{ccc}
 & & \mathbb{R}^m / \mathbb{R} \\
 & \nearrow^{\text{ev}_{B_1}} & \\
 \mathcal{M}^{B_1} & \xrightarrow{(\text{ev} \tilde{\times} \kappa)_1} & (\mathbb{R}^m \times \mathbb{R}^{n-1}) / \mathbb{R} \\
 & \searrow_{\kappa_{B_1}} & \\
 & & \mathbb{R}^{n-1} / \mathbb{R} \cong \text{Conf}(D_{n+1}).
 \end{array}$$

Similar remarks apply to curves in Σ_2 .

Definition 5.23. *With notation as in Definition 5.5, define the moduli space of cross-matched polygons in the homology classes B_1 and B_2 with sources S_1 , S_2 , T_1 and T_2 and Reeb chords (ρ_1, \dots, ρ_m) (for $m \geq 1$),*

$$\mathcal{X}\mathcal{M}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1, \mathbf{x}_2 \# \mathbf{y}_2; \rho_1, \dots, \rho_m; S_1, T_1, S_2, T_2),$$

to consist of quadruples of pseudo-holomorphic curves

$$\begin{aligned} u_1: S_1 &\rightarrow \Sigma_1 \times [0, 1] \times \mathbb{R} & u_2: T_1 &\rightarrow \Sigma_1 \times [0, 1] \times \mathbb{R} \\ v_1: S_2 &\rightarrow \Sigma_2 \times [0, 1] \times \mathbb{R} & v_2: T_2 &\rightarrow \Sigma_2 \times [0, 1] \times \mathbb{R} \end{aligned}$$

satisfying:

(X-1) u_1 represents a homology class

$$B_1^1 \in \pi_2(\mathbf{x}', \eta^{\dot{y}_{n_1-1} < \dot{y}_{n_1}}, \dots, \eta^{\dot{y}_1 < \dot{y}_2}, \mathbf{x}_1);$$

for some $1 \leq n_1 \leq n$ and $\mathbf{x}' \in \mathfrak{S}(\boldsymbol{\alpha}_1, \boldsymbol{\beta}^{\dot{y}_{n_1}})$ (so u_1 is an $n_1 + 1$ -gon); and u_2 represents a homology class

$$B_1^2 \in \pi_2(\mathbf{x}_2, \eta^{\dot{y}_{n-1} < \dot{y}_n}, \dots, \eta^{\dot{y}_{n_2} < \dot{y}_{n_1+1}}, \mathbf{x}')$$

(so u_2 is an $n_2 + 1$ -gon with $n_1 + n_2 = n + 1$).

(X-2) v_1 represents a homology class

$$B_2^1 \in \pi_2(\mathbf{y}', \zeta^{\dot{y}_{n_1-1} < \dot{y}_{n_1}}, \dots, \zeta^{\dot{y}_1 < \dot{y}_2}, \mathbf{y}_1),$$

for some $\mathbf{y}' \in \mathfrak{S}(\boldsymbol{\alpha}_2, \boldsymbol{\beta}^{\dot{y}_{n_1}})$ and v_2 represents a homology class

$$B_2^2 \in \pi_2(\mathbf{y}_2, \zeta^{\dot{y}_{n-1} < \dot{y}_n}, \dots, \zeta^{\dot{y}_{n_1} < \dot{y}_{n_1+1}}, \mathbf{y}').$$

(X-3) $B_1^1 + B_1^2 = B_1$, and $B_2^1 + B_2^2 = B_2$.

(X-4) u_2 and v_1 are provincial (i.e., they have no Reeb chords at $e\infty$).

(X-5) $\text{ev}_{B_1^1}(u_1)$ and $\text{ev}_{B_2^2}(v_1^2)$ agree up to overall translation (i.e., as elements of \mathbb{R}^m/\mathbb{R}).

(X-6) The conformal structures of the bases of u_1 and v_1 coincide, as do the conformal structures of the bases of u_2 and v_2 ; i.e.,

$$\begin{aligned} \kappa_{B_1^1}(u_1) &= \kappa_{B_2^1}(v_1) \\ \kappa_{B_1^2}(u_2) &= \kappa_{B_2^2}(v_2). \end{aligned}$$

The moduli space of embedded cross-matched polygons $\mathcal{XM}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1, \mathbf{x}_2 \# \mathbf{y}_2)$ is the union of the spaces

$$\mathcal{XM}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1, \mathbf{x}_2 \# \mathbf{y}_2; \rho_1, \dots, \rho_m; S_1, T_1, S_2, T_2),$$

over all sequences of Reeb chords and sources S_1, T_1, S_2, T_2 with

$$\chi(S_1 \natural T_1 \natural S_2 \natural T_2) = \chi_{\text{emb}}(B_1 \natural B_2).$$

(As usual, \natural denotes gluing at the corresponding punctures.) In this union, we also include the matched polygons (from Definition 5.5) with no Reeb chords, corresponding to the degenerate case $m = 0$.

Fix a cross-matched holomorphic curve $(u_1 * u_2, v_1 * v_2)$. Let $\dim(u_1)$ denote the expected dimension of the moduli space of holomorphic curves near u_1 ; i.e., $\dim(u_1) = \text{ind}(u_1) + n_1 - 2$. (Note that in the degenerate case that u_1 is an \mathbb{R} -invariant bigon, $\dim(u_1) = -1$ while the moduli space is, in fact, 0-dimensional.) Define $\dim(u_2)$, $\dim(v_1)$ and $\dim(v_2)$ similarly.

Lemma 5.24. *With respect to a generic family of almost-complex structures, the moduli spaces of cross-matched holomorphic polygons are transversally cut out. Fix a cross-matched polygon $(u_1 * u_2, v_1 * v_2)$, and suppose that c of $\{u_1, u_2, v_1, v_2\}$ are \mathbb{R} -invariant bigons (disjoint*

unions of trivial strips). Then, near $(u_1 * u_2, v_1 * v_2)$ the moduli space of cross-matched polygons has dimension

$$\dim_{XM}(u_1 * u_2, v_1 * v_2) = \dim(u_1) + \dim(u_2) + \dim(v_1) + \dim(v_2) + 3 - n + c - \min\{m - 1, 0\},$$

where m denotes the number of Reeb chords in u_1 (or equivalently v_2).

Proof. Again, the transversality statement follows from standard techniques. For the statement about dimensions, observe that we are matching the conformal structures on u_1 and v_1 , which gives an $(n_1 - 2)$ -dimensional constraint; then we are matching conformal structures on u_2 and v_2 , which gives a further $(n_2 - 2)$ -dimensional constraint; finally, we have an $(m - 1)$ -dimensional constraint coming from the matching condition on the chords (if $m \geq 1$). Together with the observation that $n = n_1 + n_2 - 1$, this explains all of the terms in the formula except c , which comes from the fact that if u_1 , say, is an \mathbb{R} -invariant bigon then $\dim(u_1)$ differs by 1 from the actual dimension of the moduli space near u_1 (because u_1 has \mathbb{R} as a stabilizer). \square

Counting cross-matched polygons also gives a chain complex $\mathcal{C}_{[\infty]}$. In fact:

Lemma 5.25. *For all sufficiently large t , the differential in $\mathcal{C}_{[t]}$ coincides with the differential counting cross-matched polygons.*

Proof. This follows from compactness and gluing arguments, similar to (but easier than) the proof of [LOT08, Proposition 9.40]. Fix a rigid, non-provincial homology class, so the pair of curves are asymptotic to at least one Reeb chord at $e\infty$: the provincial case is trivial. The compactness theorem for pseudoholomorphic combs [LOT08, Proposition 5.24], or rather its obvious extension to polygons, implies that any sequence of t -slid matched polygons with $t \rightarrow \infty$ has a subsequence converging to some pair U, V of holomorphic combs. Further, the evaluation maps ev at the (far) east punctures of U and V must satisfy that the evaluation map on the i^{th} story of U matches with the evaluation map on the $(i - 1)^{\text{st}}$ story of V , up to an overall translation on each story, while the conformal structures (i.e., the forgetful maps κ) on the i^{th} story of U and the i^{th} story of V must agree. In particular, each of U and V must have at least two stories.

Next, a dimension count implies that each of U and V has no components at $e\infty$, and consists of exactly two stories. The fact that in the limit we consider the conformal structures and evaluation maps separately means that number of matching conditions has decreased by 1, i.e., the expected dimension has increased by 1. However, each story of U or V beyond the first reduces the expected dimension by 1 on each side, so 2 overall, but only decreases the number of matching conditions imposed by the evaluation maps ($\text{ev} \tilde{\times} \kappa$) by 1. Thus, with two stories the expected dimension of the limit object is at most the same as for a 1-story t -slid matched curves; with three stories the expected dimension is at least one smaller than for a 1-story t -slid matched curve; and so on. So, for generic almost-complex structures, U and V must have exactly two stories. Degenerating a curve at $e\infty$ or having two levels of Reeb chords collapse reduces the expected dimension of the moduli space on each side by at least 1, but only reduces the number of matching conditions imposed (cf. [LOT08, Theorem 5.61]) by at most one, so is a codimension-1 degeneration overall. Thus, each of U and V must consist of two stories with no components at $e\infty$.

So, we have shown that each pair (U, V) occurring as a $t \rightarrow \infty$ end of the moduli space of t -slid matched polygons is a cross-matched polygon. Finally, the polygon analogue of the gluing result [LOT08, Proposition 5.30] implies that near each cross-matched polygon the

space $\bigcup_{t \in (T, \infty)} \mathcal{MM}_{[t]}^{B_1 \# B_2}$ is an open interval, so in particular the modulo 2 count of t -slid matched polygons and cross-matched polygons agrees. \square

5.5. Small ϵ approximation and chord-matched polygon pairs. We next consider what happens when we take the approximation parameter $\epsilon \rightarrow 0$. Our main goal is to show that, for ϵ sufficiently small, rigid cross-matched polygons correspond to rigid chord-matched polygon pairs (Definition 5.29); see Proposition 5.31. To prove this, we introduce an auxiliary notion, simplified cross-matched polygons, which are the Gromov limits of cross-matched polygons as $\epsilon \rightarrow 0$. (The definition of simplified cross-matched polygons can be refined, but Definition 5.26 will suffice for our purposes: its main role is to restrict what kinds of cross-matched polygons exist for small ϵ .)

Definition 5.26. Fix $i < i'$ and $j < j'$, and let

$$\begin{aligned} \mathbf{x}_1 &\in \mathfrak{S}(\boldsymbol{\alpha}^1, \boldsymbol{\beta}^i), & \mathbf{x}_2 &\in \mathfrak{S}(\boldsymbol{\alpha}^1, \boldsymbol{\gamma}^{i'}), \\ \mathbf{y}_1 &\in \mathfrak{S}(\boldsymbol{\alpha}^2, \boldsymbol{\beta}^j), & \mathbf{y}_2 &\in \mathfrak{S}(\boldsymbol{\alpha}^2, \boldsymbol{\gamma}^{j'}) \end{aligned}$$

be generators with the property that \mathbf{x}_1 and \mathbf{y}_1 is a complementary pair of generators, and \mathbf{x}_2 and \mathbf{y}_2 is a complementary pair of generators. Fix sequences $i = i_0 < \dots < i_{n_1} = i'$ and $j = j_0 < \dots < j_{n_2} = j'$ and homology classes

$$\begin{aligned} B_1 &\in \pi_2(\mathbf{x}_2, \eta^{i_{n_1}-1 < i_{n_1}}, \dots, \eta^{i_1 < i_2}, \mathbf{x}_1) \\ B_2 &\in \pi_2(\mathbf{y}_2, \zeta^{j_{n_2}-1 < j_{n_2}}, \dots, \zeta^{j_1 < j_2}, \mathbf{y}_1). \end{aligned}$$

Fix also punctured Riemann surfaces with boundary S_1 , S_2 , T_1 and T_2 and a sequence of Reeb chords (ρ_1, \dots, ρ_m) .

Define the moduli space of simplified cross-matched polygons in the homology classes B_1 and B_2 with sources S_1 , S_2 , T_1 and T_2 ,

$$\mathcal{SM}^{B_1 \sharp B_2}(\mathbf{x}_1 \# \mathbf{y}_1; \mathbf{x}_2 \# \mathbf{y}_2; \rho_1, \dots, \rho_m; S_1, T_1, S_2, T_2),$$

to consist of quadruples of pseudo-holomorphic curves

$$\begin{aligned} u_1: S_1 &\rightarrow \Sigma_1 \times [0, 1] \times \mathbb{R} & u_2: T_1 &\rightarrow \Sigma_1 \times [0, 1] \times \mathbb{R} \\ v_1: S_2 &\rightarrow \Sigma_2 \times [0, 1] \times \mathbb{R} & v_2: T_2 &\rightarrow \Sigma_2 \times [0, 1] \times \mathbb{R} \end{aligned}$$

such that

(SX-1) u_1 represents some

$$B_1^1 \in \pi_2(\mathbf{x}', \eta^{i_{\ell_1}-1 < i_{\ell_1}}, \dots, \eta^{i_1 < i_2}, \mathbf{x}_1)$$

for some $1 \leq \ell_1 \leq n_1$ and $\mathbf{x}' \in \mathfrak{S}(\boldsymbol{\alpha}_1, \boldsymbol{\beta}^{i_{\ell_1}})$ (so B_1^1 is an $\ell_1 + 1$ -gon); and u_2 represents some

$$B_1^2 \in \pi_2(\mathbf{x}_2, \eta^{i_{n_1}-1 < i_{n_1}}, \dots, \eta^{i_{\ell_1} < i_{\ell_1+1}}, \mathbf{x}')$$

(so B_1^2 is an $\ell'_1 + 1$ -gon for $\ell_1 + \ell'_1 - 1 = n_1$);

(SX-2) v_1 represents some

$$B_2^1 \in \pi_2(\mathbf{y}', \zeta^{j_{\ell_2}-1 < j_{\ell_2}}, \dots, \zeta^{j_1 < j_2}, \mathbf{y}_1)$$

for some $1 \leq \ell_2 \leq n_2$ and $\mathbf{y}' \in \mathfrak{S}(\boldsymbol{\alpha}_2, \boldsymbol{\beta}^{j_{\ell_2}})$ (so B_2^1 is an $\ell_2 + 1$ -gon); and v_2 represents some

$$B_2^2 \in \pi_2(\mathbf{y}_2, \zeta^{j_{n_2}-1 < j_{n_2}}, \dots, \zeta^{j_{\ell_2} < j_{\ell_2+1}}, \mathbf{y}');$$

(so B_2^2 is an $\ell'_2 + 1$ -gon for $\ell_2 + \ell'_2 - 1 = n_2$);

- (SX-3) $B_1^1 + B_1^2 = B_1$ and $B_2^1 + B_2^2 = B_2$;
 (SX-4) u_2 and v_1 are provincial (i.e., they have no Reeb chords at $e\infty$); and
 (SX-5) $\text{ev}_{B_1^1}(u_1)$ and $\text{ev}_{B_2^2}(v_2)$ agree up to overall translation (i.e., as elements of \mathbb{R}^m/\mathbb{R}).

The moduli space of embedded simplified cross-matched polygons

$$\mathcal{SM}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1, \mathbf{x}_2 \# \mathbf{y}_2)$$

is the union of the spaces

$$\mathcal{SM}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1, \mathbf{x}_2 \# \mathbf{y}_2; \rho_1, \dots, \rho_m; S_1, T_1, S_2, T_2),$$

over all sequences of Reeb chords and sources S_1, T_1, S_2, T_2 with

$$\chi(S_1 \natural T_1 \natural S_2 \natural T_2) = \chi_{\text{emb}}(B_1 \natural B_2).$$

Again, in this union, we also include the matched polygons (from Definition 5.5) with no Reeb chords, corresponding to the degenerate case $m = 0$.

When comparing Definition 5.26 with Definition 5.23, the reader should be aware of two key differences: the Heegaard diagrams appearing in Definition 5.26 do not involve approximations to the β^i or γ^j , whereas those in Definition 5.23 do; and, in Definition 5.26, the conformal moduli of the polygons are unconstrained whereas in Definition 5.23, there is a restriction coming from Property (X-6).

Lemma 5.27. *With respect to a generic family of almost-complex structures, the moduli spaces of simplified cross-matched holomorphic polygons are transversally cut out. Fix a generic $\{J_j\}$ and a simplified cross-matched polygon $(u_1 * u_2, v_1 * v_2)$, and suppose that c of $\{u_1, u_2, v_1, v_2\}$ are \mathbb{R} -invariant bigons (disjoint unions of trivial strips). Then, near $(u_1 * u_2, v_1 * v_2)$ the moduli space of cross-matched polygons has dimension*

$$\dim_{\mathcal{SM}}(u_1 * u_2, v_1 * v_2) = \dim(u_1) + \dim(u_2) + \dim(v_1) + \dim(v_2) + c - \min\{m - 1, 0\},$$

where m denotes the number of Reeb chords in u_1 (or equivalently v_2).

Proof. The proof of transversality is similar to the (omitted) proof of transversality in Lemma 5.24 (but easier, since we do not have to worry about matching the conformal structures).

For the dimension counting, note that we have $(m - 1)$ constraints coming from the Reeb chords (if m is at least one), and recall that $\dim(u_1)$, say, differs from the actual dimension of the moduli space near u_1 by 1 if u_1 is an \mathbb{R} -invariant bigon. \square

As in Definition 3.34, looking at the local multiplicities away from the isotopy region gives a map

$$\phi_\epsilon: \pi_2(\mathbf{x}_2, \eta^{\tilde{j}_{n-1} < \tilde{j}_n}, \dots, \eta^{\tilde{j}_1 < \tilde{j}_2}, \mathbf{x}_1) \rightarrow \pi_2(\mathbf{x}_2, \eta^{i_{s_{n-k-1}} < i_{s_{n-k}}}, \dots, \eta^{i_{s_1} < i_{s_2}}, \mathbf{x}_1),$$

where $k = R(i_1, \dots, i_n)$ is the number of repeated entries in the sequence i_1, \dots, i_n . As before, we will call $B_\epsilon \in \phi_\epsilon^{-1}(B')$ an *approximation to B'* .

The following is an analogue of Lemma 3.38.

Proposition 5.28. *With notation as in Definition 5.26, suppose that*

$$\begin{aligned} B_{1,\epsilon} &\in \pi_2(\mathbf{x}_2, \eta^{\tilde{j}_{n-1} < \tilde{j}_n}, \dots, \eta^{\tilde{j}_1 < \tilde{j}_2}, \mathbf{x}_1) \\ B_{2,\epsilon} &\in \pi_2(\mathbf{y}_2, \zeta^{\tilde{j}_{n-1} < \tilde{j}_n}, \dots, \zeta^{\tilde{j}_1 < \tilde{j}_2}, \mathbf{y}_1) \end{aligned}$$

are approximations to B_1 and B_2 . If $\mathcal{XM}^{B_1, \epsilon \natural B_2, \epsilon}(\mathbf{x}_1 \# \mathbf{y}_1, \mathbf{x}_2 \# \mathbf{y}_2)$ is non-empty for $\epsilon > 0$ arbitrarily small then $\mathcal{SM}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1; \mathbf{x}_2 \# \mathbf{y}_2)$ is non-empty, as well.

Proof. This follows from the same argument as Lemma 3.38, treating east ∞ as in the proof of [LOT08, Proposition 5.24]. \square

As mentioned earlier, we will never actually count the moduli space of simplified cross-matched polygons. The moduli spaces we will count (and continue to study in Section 5.6) are the moduli spaces of chord-matched polygon pairs:

Definition 5.29. *Fix generators*

$$\mathbf{x}_1 \in \mathfrak{S}(\alpha^1, \beta^i) \quad \mathbf{x}_2 \in \mathfrak{S}(\alpha^1, \beta^{i'}) \quad \mathbf{y}_1 \in \mathfrak{S}(\alpha^2, \gamma^j) \quad \mathbf{y}_2 \in \mathfrak{S}(\alpha^2, \gamma^{j'})$$

so that \mathbf{x}_1 and \mathbf{y}_1 are a complementary pair and \mathbf{x}_2 and \mathbf{y}_2 are a complementary pair. Fix homology classes $B_1 \in \pi_2(\mathbf{x}_2, \eta^{i_{n_1-1} < i_{n_1}}, \dots, \eta^{i_1 < i_2}, \mathbf{x}_1)$ and $B_2 \in \pi_2(\mathbf{y}_2, \zeta^{j_{n_2-1} < j_{n_2}}, \dots, \zeta^{j_1 < j_2}, \mathbf{y}_1)$, surfaces S_1 and S_2 , and a sequence of Reeb chords ρ_1, \dots, ρ_m . The moduli space of chord-matched polygon pairs in the homology classes B_1 and B_2 with sources S_1 and S_2 is the fibered product

$$\begin{aligned} \mathcal{N}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1, \mathbf{x}_2 \# \mathbf{y}_2; \rho_1, \dots, \rho_m; S_1, S_2) \\ = \mathcal{M}^{B_1}(\mathbf{x}_2, \eta^{i_{n_1-1} < i_{n_1}}, \dots, \eta^{i_2 < i_1}, \mathbf{x}_1; \rho_1, \dots, \rho_m; S_1) \\ \times_{\text{ev}_{B_1} = \text{ev}_{B_2}} \mathcal{M}^{B_2}(\mathbf{y}_2, \zeta^{j_{n_2-1} < j_{n_2}}, \dots, \zeta^{j_2 < j_1}, \mathbf{y}_1; \rho_1, \dots, \rho_m; S_2). \end{aligned}$$

The moduli space of embedded chord-matched polygon pairs $\mathcal{N}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1; \mathbf{x}_2 \# \mathbf{y}_2)$ is the union of the spaces

$$\mathcal{N}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1, \mathbf{x}_2 \# \mathbf{y}_2; \rho_1, \dots, \rho_m; S_1, S_2)$$

over all sequences of Reeb chords and sources S_1 and S_2 with

$$(5.30) \quad \chi(S_1) = \chi_{\text{emb}}(B_1) \quad \text{and} \quad \chi(S_2) = \chi_{\text{emb}}(B_2).$$

In this union we also include the degenerate case where there are no Reeb chords (i.e., $m = 0$) and one of B_1 or B_2 is the trivial homology class of bigons.

Note that the two polygons in Definition 5.29 typically have a different number of sides, and their conformal structures are unconstrained. The moduli space of embedded chord-matched polygon pairs has expected dimension

$$\text{ind}_{\text{emb}}(B_1 \natural B_2) + n_1 + n_2 - 3.$$

Proposition 5.31. *Fix a pair of domains $B_{1,\epsilon}$ and $B_{2,\epsilon}$ which are approximations to a pair of domains B_1 and B_2 , so that the expected dimension of polygons in the glued homology class $B_1 \natural B_2$ is 0. Then for ϵ sufficiently small, the moduli space of cross-matched polygons $\mathcal{XM}^{B_{1,\epsilon} \natural B_{2,\epsilon}}(\mathbf{x}_1 \# \mathbf{y}_1, \mathbf{x}_2 \# \mathbf{y}_2)$ in the homology class $B_{1,\epsilon} \natural B_{2,\epsilon}$ is empty except in the following cases:*

- (The non-degenerate case.) If there are $m > 0$ Reeb chords, $i_{n_1+1} = \dots = i_n$, $j_1 = \dots = j_{n_1}$, and $\phi_\epsilon(B_1^2)$ and $\phi_\epsilon(B_2^1)$ are trivial domains (all of whose multiplicities are 0). (Here, B_1^2 and B_2^1 are from items (X-1) and (X-2), respectively.)
- (The degenerate case.) If there are no Reeb chords ($m = 0$), and one of B_1 or B_2 is the trivial domain (all of whose multiplicities are 0).

Further,

$$\sum_{\substack{\phi_\epsilon(B_{1,\epsilon})=B_1 \\ \phi_\epsilon(B_{2,\epsilon})=B_2}} \# \mathcal{XM}^{B_1, \epsilon \natural B_2, \epsilon}(\mathbf{x}_1 \# \mathbf{y}_1, \mathbf{x}_2 \# \mathbf{y}_2) = \# \mathcal{N}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1, \mathbf{x}_2 \# \mathbf{y}_2).$$

Proof. This proof can be thought of as a variant of the proof of Proposition 3.52. In the degenerate case of curves with no Reeb chords ($m = 0$), the result in fact follows from the proof of Proposition 3.52, so we will now restrict attention to the case of $m > 0$ Reeb chords.

Let $\{(u_1^k * u_2^k, v_1^k * v_2^k)\}_{k=1}^\infty$ be a sequence of cross-matched polygons in the homology classes B_{1,ϵ_k}^1 , B_{1,ϵ_k}^2 , B_{2,ϵ_k}^1 and B_{2,ϵ_k}^2 with

$$\dim_{XM}(u_1^k * u_2^k, v_1^k * v_2^k) = 0$$

and perturbation parameters ϵ_k converging to 0. Here, B_{1,ϵ_k}^1 is an approximation (in the sense of Definition 3.34) to some homology class B_1^1 , and similarly for B_{1,ϵ_k}^2 , B_{2,ϵ_k}^1 and B_{2,ϵ_k}^2 . Restricting attention to a subsequence, we can assume that u_1^k and v_1^k are $(n_1 + 1)$ -gons and u_2^k and v_2^k are $(n_2 + 1)$ -gons, for n_1 and n_2 independent of k . Proposition 5.28 then guarantees that $\mathcal{SM}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1, \mathbf{x}_2 \# \mathbf{y}_2)$ is non-empty. Let $(u_1' * u_2', v_1' * v_2') \in \mathcal{SM}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1, \mathbf{x}_2 \# \mathbf{y}_2)$.

Observe that Lemma 3.36 holds even in the present case where one of the sets of attaching curves—the α 's—contains some arcs, giving:

$$\begin{aligned} \dim(u_1^k) - \dim(u_1') &= R(i_1, \dots, i_{n_1}) \\ \dim(u_2^k) - \dim(u_2') &= R(i_{n_1}, \dots, i_n) \\ \dim(v_1^k) - \dim(v_1') &= R(j_1, \dots, j_{n_1}) \\ \dim(v_2^k) - \dim(v_2') &= R(j_{n_1}, \dots, j_n). \end{aligned}$$

(The notation \dim has a slightly different meaning in this section from in Lemma 3.36: here, an \mathbb{R} -invariant u has $\dim(u) = -1$. Also, in the present setting, Case (2) of Lemma 3.36 does not occur.) It is easy to see that

$$\begin{aligned} R(i_1, \dots, i_{n_1}) + R(j_1, \dots, j_{n_1}) &= n_1 - 1 \\ R(i_{n_1}, \dots, i_n) + R(j_{n_1}, \dots, j_n) &= n_2 - 1. \end{aligned}$$

We conclude that

$$\begin{aligned} 0 \leq \dim_{SM}(u_1' * u_2', v_1' * v_2') &= \dim(u_1') + \dim(u_2') + \dim(v_1') + \dim(v_2') + c' - m + 1 \\ &= \dim(u_1^k) + \dim(u_2^k) + \dim(v_1^k) + \dim(v_2^k) + 2 \\ &\quad - n_1 - n_2 + c' - m + 1 \\ &= \dim_{XM}(u_1^k * u_2^k, v_1^k * v_2^k) - 2 + (c' - c) \\ &= (c' - c) - 2, \end{aligned}$$

where c' is the number of $\{u_1', u_2', v_1', v_2'\}$ which are \mathbb{R} -invariant bigons, and c is the number of $\{u_1^k, u_2^k, v_1^k, v_2^k\}$ which are \mathbb{R} -invariant bigons. Since there is at least one Reeb chord, u_1^k , u_2^k , v_1^k and v_2^k are not \mathbb{R} -invariant bigons, so $c' \leq 2$. Moreover, if $c' = 2$ then $i_{n_1+1} = \dots = i_n$ and $j_1 = \dots = j_{n_1}$. This proves the first half of the statement.

For the second half, Lemma 3.50 says that the forgetful maps $\kappa_{B_1^1}$ and $\kappa_{B_2^1}$ are both degree 1. So, dropping Condition (X-6) of Definition 5.23 and forgetting the components $u_{2,\epsilon}$ and $v_{1,\epsilon}$ does not change the holomorphic curve counts. Since $i_{n_1+1} = \dots = i_n$ and $j_1 = \dots = j_{n_1}$,

we have $i_1 < \dots < i_{n_1}$ and $j_{n_1} < \dots < j_n$, and so the remaining curves (u_1^k, v_2^k) can be viewed as elements of $\mathcal{N}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1, \mathbf{x}_2 \# \mathbf{y}_2)$. This completes the proof. \square

The moduli space of chord-matched polygon pairs can be used to construct a chain complex. Specifically, define $\mathcal{C}_{\mathcal{N}} = \{\mathcal{C}_{\mathcal{N}}^{i \times j}\}_{i \times j \in \mathbb{I} \times \mathbb{J}}$ to be $\mathcal{C}_{\mathcal{N}}^{i \times j} = \widehat{CF}(\alpha^1 \cup \alpha^2, \beta^i \cup \gamma^j, z)$ with differential

$$D^{i \times j < i' \times j'}(\mathbf{x}_1 \# \mathbf{y}_1) = \sum_{\substack{\mathbf{x}_2 \# \mathbf{y}_2 \in \mathfrak{S}(\alpha^1 \cup \alpha^2, \beta^{i' \times j'} \cup \gamma^{i' \times j'}) \\ i = i_1 < i_2 < \dots < i_{n_1} = i' \\ j = j_1 < j_2 < \dots < j_{n_2} = j' \\ (B_1, B_2) \text{ s.t. } \text{ind}_{\text{emb}}(B_1 \natural B_2) = 3 - n_1 - n_2}} \# \mathcal{N}^{B_1 \natural B_2}(\mathbf{x}_1 \# \mathbf{y}_1, \mathbf{x}_2 \# \mathbf{y}_2) \mathbf{x}_2 \# \mathbf{y}_2.$$

One could verify directly that $\partial^2 = 0$ on $\mathcal{C}_{\mathcal{N}}$, but this also follows from Proposition 5.31.

5.6. Time dilation for chord-matched polygon pairs. The “time dilation” argument from [LOT08, Chapter 9] identifies the chain complex $\mathcal{C}_{\mathcal{N}}$ with the chain complex for the tensor product. Before giving it, we introduce one more piece of terminology. We have avoided describing the compactifications of the moduli spaces of polygons, via *holomorphic polygonal combs*, because of the cumbersome notation. But we will need one special case of these objects:

Definition 5.32. Fix a sequence of sets of attaching circles β^0, \dots, β^n , a subsequence $i_0 = 0 < i_1 < \dots < i_m = n$, generators $\mathbf{x}_j \in \mathfrak{S}(\alpha, \beta^{i_j})$ ($j = 0, \dots, m$) and $\eta^j \in \mathfrak{S}(\beta^j, \beta^{j+1})$ ($j = 0, \dots, n-1$), and homology classes $B_j \in \pi_2(\mathbf{x}_{j+1}, \eta^{i_{j+1}}, \dots, \eta^{i_j}, \mathbf{x}_j)$. A spinal holomorphic polygonal comb is a sequence of holomorphic polygons (as in Definition 4.17) (u_1, u_2, \dots, u_m) where $u_j \in \mathcal{M}^{B_j}$. We say that this polygonal comb has m stories and represents the homology class $B_1 + \dots + B_m \in \pi_2(\mathbf{x}_m, \eta^{n-1}, \dots, \eta^0, \mathbf{x}_0)$; and u_i is the i^{th} story of the comb.

There is a trivial spinal holomorphic polygonal comb, with $m = 0$ stories. Holomorphic polygons can be viewed as 1-story spinal holomorphic polygonal combs.

The following definition is a generalization of [LOT08, Definition 9.31] to polygons:

Definition 5.33. A trimmed simple ideal-matched polygon pair connecting complementary pairs of generators $\mathbf{x}_1 \# \mathbf{y}_1$ and $\mathbf{x}_2 \# \mathbf{y}_2$ in the homology classes

$$B_1 \in \pi_2(\mathbf{x}_2, \eta^{i_{n-1} < i_n}, \dots, \eta^{i_1 < i_2}, \mathbf{x}_1) \text{ and } B_2 \in \pi_2(\mathbf{y}_2, \eta^{j_{m-1} < j_m}, \dots, \zeta^{j_1 < j_2}, \mathbf{y}_1)$$

is a pair of spinal holomorphic polygonal combs w_1 and w_2 where

- (T-A) One of w_1 or w_2 is trivial and the other is a rigid (i.e., index $3 - c$ where c is the number of corners) holomorphic polygon with no e punctures or
- (T-B) (w_1, w_2) has the following properties:
 - (T-B1) The comb w_1 is a (one story) holomorphic curve representing B_1 which is asymptotic to the non-trivial sequence of non-empty sets of Reeb chords $\vec{\rho} = (\rho_1, \dots, \rho_q)$.
 - (T-B2) The curve w_1 is rigid (with respect to $\vec{\rho}$).
 - (T-B3) The comb w_2 is a q -story spinal holomorphic polygonal building representing the homology class B_2 .
 - (T-B4) Each story of w_2 is rigid.
 - (T-B5) Each of w_1 and w_2 is strongly boundary monotone.

- (T-B6) For each $i = 1, \dots, q$, the east punctures of the i^{th} story of w_2 are labeled, in order, by a non-empty sequence of Reeb chords $(-\rho_1^i, \dots, -\rho_\ell^i)$ which have the property that the sequence of singleton sets of chords $\bar{\rho}^i = (\{\rho_1^i\}, \dots, \{\rho_\ell^i\})$ is composable.
- (T-B7) The composition of the sequence of singleton sets of Reeb chords on the i^{th} story of w_2 (with reversed orientation) coincides with the i^{th} set of Reeb chords $\boldsymbol{\rho}_i$ appearing on the boundary of w_1 .

We can define a chain complex \mathcal{C}_{tsic} , which counts points in zero-dimensional moduli spaces of trimmed simple ideal-matched polygon pairs. Rather than proving directly that this does in fact define a chain complex, we identify it (up to homotopy equivalence) with the chain complex $\mathcal{C}_{\mathcal{N}}$:

Proposition 5.34. *The chain complex whose differential counts chord-matched polygon pairs $\mathcal{C}_{\mathcal{N}}$ is homotopy equivalent to the complex whose differential counts trimmed simple ideal-matched polygon pairs \mathcal{C}_{tsic} .*

Proof. The proof of this proposition follows most of the time dilation proof of the pairing theorem for \widehat{HF} [LOT08, Chapter 9]. In words, we consider yet another chain complex which counts chord-matched polygon pairs where now the matching condition on the chords is further perturbed by scaling out by a parameter T ; i.e., $T \text{ev}_{B_1}(u) = \text{ev}_{B_2}(v)$. There are chain homotopy equivalences between these complexes as we vary the parameter T . (See [LOT08, Proposition 9.22].) The novelty in the present application of this argument is that now, there can be canceling ends in the moduli spaces of polygons which correspond to polygons connecting the various β^i ; but this came up already in the proof of Lemma 5.22 above.

Next, we make the parameter T very large. For large T , counts in the moduli spaces of T -matched polygons stabilize to counts of trimmed simple ideal-matched polygon pairs, according to the argument from [LOT08, Proposition 9.40]. \square

Now we put together the above steps to prove Theorem 5:

Proof of Theorem 5. We identify

$$\widehat{\mathbf{CF}}(\Sigma_1 \cup \Sigma_2, \boldsymbol{\alpha}^1 \cup \boldsymbol{\alpha}^2, (\{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i < i'}\}_{i, i' \in \mathbb{I}}) \# (\{\gamma^j\}_{j \in \mathbb{J}}, \{\zeta^{j < j'}\}_{j, j' \in \mathbb{J}}), z)$$

up to homotopy equivalence with the chain complex counting trimmed simple ideal-matched polygon pairs \mathcal{C}_{tsic} , by applying, in turn, Proposition 5.13, Lemma 5.22, Lemma 5.25, and Proposition 5.31. The differential in \mathcal{C}_{tsic} is identified with the differential in the tensor product complex

$$\widehat{\mathbf{CFA}}(\Sigma_1, \boldsymbol{\alpha}_1, \{\beta^i\}, z) \boxtimes \widehat{\mathbf{CFD}}(\Sigma_2, \boldsymbol{\alpha}_2, \{\gamma^j\}, z).$$

This follows from the expression for $D^{i \times j < i' \times j'}$ from Equation (2.7): the string of operations on the type D side counts k -story spinal holomorphic polygonal combs in Σ_2 and the node $F^{i \leq i'}$ pairs these with corresponding holomorphic polygons in Σ_1 (with the understanding that the terms in δ_j and $F^{i \leq i'}$ when $i = i'$ count bigons). \square

5.7. On boundedness. Suppose that both $(\Sigma, \boldsymbol{\alpha}^1, \{\beta^i\}_{i \in \mathbb{I}}, z)$ and $(\Sigma, \boldsymbol{\alpha}^2, \{\gamma^i\}_{i \in \mathbb{I}}, z)$ are provincially admissible, but neither is admissible. Then, a variant of Theorem 5 remains true:

Theorem 6. *Let (Σ_1, α^1, z) and (Σ_2, α^2, z) be surfaces-with-boundary, each equipped with complete sets of bordered attaching curves α^1 and α^2 and basepoints $z \in \partial\Sigma_i$ with $\partial(\Sigma_1, \alpha^1) = \mathcal{Z}$ and $\partial(\Sigma_2, \alpha^2) = -\mathcal{Z}$. Let*

$$(\Sigma_1, \mathbb{I}, \{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i < i'}\}_{i, i' \in \mathbb{I}}) \text{ and } (\Sigma_2, \mathbb{J}, \{\gamma^j\}_{j \in \mathbb{J}}, \{\zeta^{j < j'}\}_{j, j' \in \mathbb{J}})$$

be chain complexes of attaching circles in Σ_1 and Σ_2 . respectively. Suppose that both of the multi-diagrams $(\Sigma_1, \alpha^1, \{\beta^i\}_{i \in \mathbb{I}}, z)$ and $(\Sigma_2, \alpha^2, \{\gamma^j\}_{j \in \mathbb{J}}, z)$ are provincially admissible. Let M be a module with

$$M \simeq \widehat{\mathbf{CFA}}(\Sigma_1, \alpha^1, \{\beta^i\}, \{\eta^{i < i'}\}, z)$$

and such that M is bounded. Then, there is an $\mathbb{I} \times \mathbb{J}$ -filtered quasi-isomorphism

$$\begin{aligned} M \boxtimes \widehat{\mathbf{CFD}}(\Sigma_2, \alpha^2, \{\gamma^j\}, \{\zeta^{j < j'}\}, z) \\ \simeq \widehat{\mathbf{CF}}(\Sigma_1 \cup \Sigma_2, \alpha, (\{\beta^i\}, \{\eta^{i < i'}\}) \# (\{\gamma^j\}, \{\zeta^{j < j'}\}), z), \end{aligned}$$

where α is an isotopic copy of $\alpha^1 \cup \alpha^2$, chosen so that the second multi-diagram is admissible.

Proof. There is an isotopic copy ξ^2 of α^2 so that $(\Sigma, \xi^2, \{\gamma^j\}_{j \in \mathbb{J}}, z)$ is admissible (see [LOT08, Proposition 4.25]). The isotopy induces a filtered chain homotopy equivalence

$$\widehat{\mathbf{CFD}}(\Sigma_2, \alpha^2, \{\gamma^j\}, \{\zeta^{j < j'}\}, z) \simeq \widehat{\mathbf{CFD}}(\Sigma_2, \xi^2, \{\gamma^j\}, \{\zeta^{j < j'}\}, z),$$

with $\widehat{\mathbf{CFD}}(\Sigma_2, \xi^2, \{\gamma^j\}, \{\zeta^{j < j'}\}, z)$ bounded. This in turn gives the first of the following homotopy equivalences:

$$\begin{aligned} M \boxtimes \widehat{\mathbf{CFD}}(\Sigma_2, \alpha^2, \{\gamma^j\}, \{\zeta^{j < j'}\}, z) \\ \simeq M \boxtimes \widehat{\mathbf{CFD}}(\Sigma_2, \xi^2, \{\gamma^j\}, \{\zeta^{j < j'}\}, z) \\ \simeq \widehat{\mathbf{CFA}}(\Sigma_1, \alpha^1, \{\beta^i\}, \{\eta^{i < i'}\}, z) \boxtimes \widehat{\mathbf{CFD}}(\Sigma_2, \xi^2, \{\gamma^j\}, \{\zeta^{j < j'}\}, z) \\ \simeq \widehat{\mathbf{CF}}(\Sigma_1 \cup \Sigma_2, \alpha^1 \cup \xi^2, (\{\beta^i\}, \{\eta^{i < i'}\}) \# (\{\gamma^j\}, \{\zeta^{j < j'}\}), z) \\ \simeq \widehat{\mathbf{CF}}(\Sigma_1 \cup \Sigma_2, \alpha, (\{\beta^i\}, \{\eta^{i < i'}\}) \# (\{\gamma^j\}, \{\zeta^{j < j'}\}), z). \end{aligned}$$

The third homotopy equivalence above is Theorem 5, and the fourth is induced by the isotopies from $\alpha^1 \cup \xi^2$ to α (Proposition 3.30). \square

The bounded models M needed in Theorem 6 can be constructed either geometrically, by isotoping the α curves (as in the above proof) or by more algebraic considerations. One more algebraic approach is to form $\widehat{\mathbf{CFA}}(\Sigma_1, \alpha, \{\beta^i\}, \{\eta^{i < i'}\}, z) \boxtimes^{\mathcal{A}} \text{Bar}^{\mathcal{A}} \boxtimes_{\mathcal{A}} \mathcal{A}_{\mathcal{A}}$ (see [LOT15, Section 2.3.3]). Another more finite construction is to use a combinatorially describable, bounded model for $\widehat{\mathbf{CFDA}}(\mathbb{I})$, for example using [LOT15, Proposition 9.2].

5.8. The pairing theorem for bimodules. We turn next to the bimodule analogue of Theorem 5:

Theorem 7. *Let $(\Sigma_1, \alpha^1, \mathbf{z})$ (respectively $(\Sigma_2, \alpha^2, \mathbf{z})$) be a surface with two boundary components, equipped with a complete sets of bordered attaching curves α^1 (respectively α^2), compatible with \mathcal{Z}_0 and \mathcal{Z} (respectively $-\mathcal{Z}$ and \mathcal{Z}_2), for some $\mathcal{Z}_0, \mathcal{Z}$ and \mathcal{Z}_2 . Let*

$$(\Sigma_1, \mathbb{I}, \{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i_1 < i_2}\}_{i_1, i_2 \in \mathbb{I}}) \text{ and } (\Sigma_2, \mathbb{J}, \{\gamma^j\}_{j \in \mathbb{J}}, \{\zeta^{j_1 < j_2}\}_{j_1, j_2 \in \mathbb{J}})$$

be chain complexes of attaching circles in Σ_1 and Σ_2 respectively. Suppose that both $\mathcal{H}_1 = (\Sigma_1, \alpha, \{\beta^i\}_{i \in \mathbb{I}}, \mathbf{z})$ and $\mathcal{H}_2 = (\Sigma_2, \alpha, \{\gamma^j\}_{j \in \mathbb{J}}, \mathbf{z})$ are provincially admissible, and either \mathcal{H}_1 is right-admissible or \mathcal{H}_2 is left-admissible. Then, there is an $\mathbb{I} \times \mathbb{J}$ -filtered quasi-isomorphism

$$\begin{aligned} {}^{\mathcal{A}_L} \widehat{\mathbf{CFDA}}(\Sigma_1, \alpha^1, \{\beta^i\}, \{\eta^{i_1 < i_2}\}, \mathbf{z}) \boxtimes_{\mathcal{A}(\mathcal{Z})} \widehat{\mathbf{CFDA}}(\Sigma_2, \alpha^2, \{\gamma^j\}, \{\zeta^{j_1 < j_2}\}, \mathbf{z})_{\mathcal{A}_R} \\ \simeq {}^{\mathcal{A}_L} \widehat{\mathbf{CFDA}}(\Sigma_1 \cup \Sigma_2, \alpha^1 \cup \alpha^2, (\{\beta^i\}, \{\eta^{i_1 < i_2}\}) \# (\{\gamma^j\}, \{\zeta^{j_1 < j_2}\}), \mathbf{z})_{\mathcal{A}_R}, \end{aligned}$$

where, $\mathcal{A}_L = \mathcal{A}(-\partial_L \Sigma_1)$ and $\mathcal{A}_R = \mathcal{A}(\partial_R \Sigma_2)$.

Proof. The proof is similar to the proof of Theorem 5. We review this proof in outline, indicating the changes necessary for the bimodule case:

- (1) In the same vein as Proposition 5.2, the admissibility conditions guarantee that the glued diagram is admissible. (See [LOT15, Lemma 5.24].)
- (2) Glue Σ_1 and Σ_2 , identifying $\partial_R \Sigma_1$ and $\partial_L \Sigma_2$. The straightforward generalization of Proposition 5.13 identifies holomorphic curves in $\Sigma_1 \cup \Sigma_2$ with pairs of matched polygons u and v in Σ_1 and Σ_2 , respectively. In the bimodule case, the chords along $\partial_R u$ are matched with chords in $\partial_L v$, and some chords in $\partial_R v$ are constrained to lie at the same heights. The analogue of $\mathcal{C}_{[0]}$ is a filtered type DA bimodule.
- (3) Consider the analogue of t -matched curves, where now the heights of the chords in u which map to $\partial_R \Sigma_1$ are translated as compared with the heights of chords in v which map to $\partial_L \Sigma_2$. Counting these curves gives the DA bimodule operations on the generalization of $\mathcal{C}_{[t]}$. The filtered DA quasi-isomorphism type is independent of t .
- (4) Send t to infinity as before. The appropriate generalizations of cross-matched polygons which appear in the $t \rightarrow \infty$ limit are pairs of polygons $u_1 * u_2$ and $v_1 * v_2$ with the following properties:
 - (a) The conformal moduli of the polygons underlying u_1 and v_1 are matched, as are the conformal moduli of u_2 and v_2 .
 - (b) Relative heights of the chords in u_1 along $\partial_R \Sigma_1$ are matched with relative heights of the chords in v_2 along $\partial_L \Sigma_2$. Rather than being required to be provincial, the polygons u_2 and v_1 are required to be disjoint from $\partial_R \Sigma_1$ and $\partial_L \Sigma_2$.
 - (c) Heights of chords in $v_1 * v_2$ appearing along $\partial_R \Sigma_2$ satisfy the constraints dictated by the action of $\mathcal{A}(\mathcal{Z}_R)$.
- (5) Make the approximation parameter ϵ sufficiently small. According to the analogue of Proposition 5.31, cross-matched polygons now correspond to chord-matched polygon pairs, analogous to Definition 5.29, where once again the chord matching occurs along the interface between Σ_1 and Σ_2 .
- (6) Dilate time along the interface between Σ_1 and Σ_2 , so chord-matched polygon pairs converge to trimmed simple ideal-matched polygonal pairs.
- (7) The resulting curve counts are identified now with the tensor product of the two filtered DA bimodules.

With these modifications, the proof is now complete. \square

We have the following variant of Theorem 7 with weaker admissibility hypotheses:

Theorem 8. *Let $(\Sigma_1, \alpha^1, \mathbf{z})$ (respectively $(\Sigma_2, \alpha^2, \mathbf{z})$) be a surface with two boundary components, equipped with a complete set of bordered attaching curves α^1 (respectively α^2) compatible with \mathcal{Z}_0 and \mathcal{Z} (respectively $-\mathcal{Z}$ and \mathcal{Z}_2). Let*

$$(\Sigma_1, \mathbb{I}, \{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i < i'}\}_{i, i' \in \mathbb{I}}) \text{ and } (\Sigma_2, \mathbb{J}, \{\gamma^j\}_{j \in \mathbb{J}}, \{\zeta^{j < j'}\}_{j, j' \in \mathbb{J}})$$

be chain complexes of attaching circles in Σ_1 and Σ_2 , respectively. Suppose that both of the multi-diagrams $(\Sigma_1, \alpha, \{\beta^i\}_{i \in \mathbb{I}}, \mathbf{z})$ $(\Sigma_2, \alpha, \{\gamma^j\}_{j \in \mathbb{J}}, \mathbf{z})$ are provincially admissible. Let M be a module with

$$M \simeq \widehat{\mathbf{CFDA}}(\Sigma_1, \alpha^1, \{\beta^i\}, \{\eta^{i < i'}\}, \mathbf{z})$$

and such that M is bounded. Then, there is an $\mathbb{I} \times \mathbb{J}$ -filtered quasi-isomorphism

$$\begin{aligned} M \boxtimes \widehat{\mathbf{CFDA}}(\Sigma_2, \alpha^2, \{\gamma^j\}, \{\zeta^{j < j'}\}, \mathbf{z}) \\ \simeq \widehat{\mathbf{CFDA}}(\Sigma_1 \cup \Sigma_2, \alpha, (\{\beta^i\}, \{\eta^{i < i'}\}) \# (\{\gamma^j\}, \{\zeta^{j < j'}\}), \mathbf{z}), \end{aligned}$$

where α is an isotopic copy of $\alpha^1 \cup \alpha^2$, chosen so that the second multi-diagram is admissible.

Proof. This follows from Theorem 7 exactly as Theorem 6 follows from Theorem 5. \square

5.9. The pairing theorem for triangles. There is a somewhat simpler statement of the pairing theorem for triangles.

Let $\mathcal{H}_L = (\Sigma_L, \alpha_L, \beta_L, \mathbf{z}_L)$ be a bordered Heegaard diagram with two boundary components, and let $\mathcal{H}_R = (\Sigma_R, \alpha_R, \beta_R^0, \beta_R^1, \mathbf{z}_R)$ be a bordered Heegaard triple with two boundary components, with $\partial_R \mathcal{H}_L = \partial_L \mathcal{H}_R$. Choose a cycle $\eta^{0 < 1} \in \widehat{CF}(\beta_R^0, \beta_R^1, \mathbf{z}_R)$. Let β'_L be an approximation to β_L , and let $\Theta \in \widehat{CF}(\beta_L, \beta'_L, \mathbf{z}_L)$ be a cycle generating the top-dimensional homology group. Abbreviate

$$\mathcal{H}'_L = (\Sigma_L, \alpha_L, \beta'_L, \mathbf{z}_L) \quad \mathcal{H}_R^0 = (\Sigma_R, \alpha_R, \beta_R^0, \mathbf{z}_R) \quad \mathcal{H}_R^1 = (\Sigma_R, \alpha_R, \beta_R^1, \mathbf{z}_R).$$

Assume that both of \mathcal{H}_L and \mathcal{H}_R are provincially admissible and that either \mathcal{H}_L is right-admissible or \mathcal{H}_R is left-admissible.

Proposition 5.35. *There is a homotopy-commutative square*

$$(5.36) \quad \begin{array}{ccc} \widehat{CFDA}(\mathcal{H}_L \cup \mathcal{H}_R^0) & \xrightarrow{m_2(\Theta \otimes \eta^{0 < 1}, \cdot)} & \widehat{CFDA}(\mathcal{H}'_L \cup \mathcal{H}_R^1) \\ \downarrow & & \downarrow \\ \widehat{CFDA}(\mathcal{H}_L) \boxtimes \widehat{CFDA}(\mathcal{H}_R^0) & \xrightarrow{m_2(\Theta, \cdot) \boxtimes m_2(\eta^{0 < 1}, \cdot)} & \widehat{CFDA}(\mathcal{H}'_L) \boxtimes \widehat{CFDA}(\mathcal{H}_R^1), \end{array}$$

where the vertical maps are induced by the pairing theorem for bigons ([LOT08, Theorem 1.3]). Analogous statements hold for pairing AA bimodules with DA or DD bimodules, DA bimodules with DD bimodules, and DD bimodules with AA bimodules; or for pairing bimodules with modules.

Proof. This follows from the pairing theorem, Theorem 7. Consider the 1-step chain complex $(\{\beta_L\}, \cdot)$ in Σ_L and the 2-step chain complex $(\{\beta_R^0, \beta_R^1\}, \eta^{0 < 1})$ in Σ_R . Theorem 7 gives a $\{0, 1\}$ -filtered homotopy equivalence

$$\begin{aligned} \widehat{CFDA}(\alpha_L, \beta_L, \mathbf{z}_L) \boxtimes \widehat{\mathbf{CFDA}}(\alpha_R, \{\beta_R^0, \beta_R^1\}, \{\eta^{0 < 1}\}, \mathbf{z}_R) \\ \simeq \widehat{\mathbf{CFDA}}(\alpha_L \cup \alpha_R, \{\beta_L \cup \beta_R^0, \beta_L \cup \beta_R^1\}, \{\Theta \otimes \epsilon_{0 < 1}\}, \mathbf{z}_L \cup \mathbf{z}_R). \end{aligned}$$

Unpacking the definitions, the right-hand side is exactly the top row of Diagram (5.36), and the left-hand side is the bottom row of Diagram (5.36). The homotopy equivalence furnishes the vertical arrows, as well as a diagonal arrow $\widehat{CFDA}(\mathcal{H}_L \cup \mathcal{H}_R^0) \rightarrow \widehat{CFDA}(\mathcal{H}'_L) \boxtimes \widehat{CFDA}(\mathcal{H}_R^1)$, which is the homotopy in “homotopy-commutative.” \square

5.10. **The exact triangles agree.** Let K be a framed knot in a 3-manifold Y , and let $Y_r(K)$ denote r -surgery on K . In [OSz04a], an exact triangle

$$(5.37) \quad \begin{array}{ccc} \widehat{HF}(Y_\infty(K)) & \longrightarrow & \widehat{HF}(Y_{-1}(K)) \\ & \nwarrow & \swarrow \\ & \widehat{HF}(Y_0(K)) & \end{array}$$

was constructed. The maps in (5.37) were defined by counting holomorphic triangles.

In [LOT08, Chapter 11] we gave another construction of an exact triangle of the form (5.37) by explicitly writing down maps between three different framed solid tori and invoking the pairing theorem. This was generalized slightly in [LOT14a, Theorem 5] to the case that Y is a bordered 3-manifold with two boundary components (and K lies in the interior of Y).

In fact, both constructions prove slightly more: they give quasi-isomorphisms

$$(5.38) \quad \widehat{CF}(Y_{-1}(K)) \simeq \text{Cone}(\theta: \widehat{CF}(Y_0(K)) \rightarrow \widehat{CF}(Y_\infty(K)))$$

or

$$(5.39) \quad \widehat{CFDD}(Y_{-1}(K)) \simeq \text{Cone}(\theta: \widehat{CFDD}(Y_0(K)) \rightarrow \widehat{CFDD}(Y_\infty(K)))$$

in the closed or bordered cases, respectively. Again, the map θ is defined in [OSz04a] by counting holomorphic triangles and in [LOT08, LOT14a] by an explicit formula together with the pairing theorem.

The goal of this section is to identify the two surgery exact triangles. We will prove:

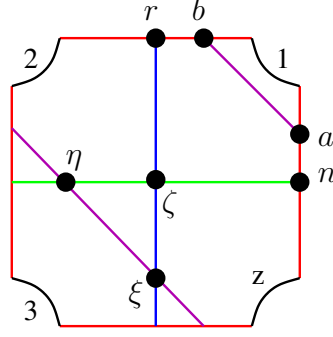
Proposition 5.40. *The map θ in Formula (5.39), as defined in [LOT14a, Theorem 5], is given up to homotopy by counting holomorphic triangles in a suitable bordered Heegaard triple-diagram. More precisely, with notation as in Sections 5.10.1 and 5.10.2, there is a homotopy-commutative square*

$$\begin{array}{ccc} \widehat{CFDD}(\mathcal{H} \cup \mathcal{H}_0) & \xrightarrow{\theta_{OS}} & \widehat{CFDD}(\mathcal{H} \cup \mathcal{H}_\infty) \\ \simeq \downarrow & & \downarrow \simeq \\ \widehat{CFDDA}(\mathcal{H}) \boxtimes \widehat{CFD}(\mathcal{H}_0) & \xrightarrow{\theta_{LOT}} & \widehat{CFDDA}(\mathcal{H}) \boxtimes \widehat{CFD}(\mathcal{H}_\infty). \end{array}$$

where the vertical arrows are induced by the pairing theorem for bigons, the map θ_{OS} is defined by counting holomorphic triangles in a bordered Heegaard triple diagram, and the map $\theta_{LOT} = \mathbb{I} \boxtimes \theta$ with θ as described in Section 5.10.1.

Analogous statements hold for type DA and AA bimodules; for D and A modules in the one-boundary-component case; and for \widehat{CF} in the closed case.

Corollary 5.41. *The surgery exact triangle constructed in [LOT08, Chapter 11] agrees with the original surgery exact triangle constructed in [OSz04a]. More precisely, there is a*

FIGURE 11. **A bordered Heegaard multi-diagram.**

homotopy-commutative square

$$\begin{array}{ccc}
 \widehat{CF}(\mathcal{H} \cup \mathcal{H}_0) & \xrightarrow{m_2(\Theta_{\gamma, \beta, \cdot})} & \widehat{CF}(\mathcal{H} \cup \mathcal{H}_\infty) \\
 \downarrow \simeq & & \downarrow \simeq \\
 \widehat{CFA}(\mathcal{H}) \boxtimes \widehat{CFD}(\mathcal{H}_0) & \xrightarrow{\mathbb{I}_{\widehat{CFA}(\mathcal{H})} \boxtimes \theta} & \widehat{CFA}(\mathcal{H}) \boxtimes \widehat{CFD}(\mathcal{H}_\infty),
 \end{array}$$

where the vertical maps are quasi-isomorphisms induced by the pairing theorem for bigons ([LOT08, Theorem 1.3]), and θ is defined as in Equation (5.42).

(This is also [LOT14a, Proposition 4.1], the proof of which was deferred to here.)

To prove Proposition 5.40 and Corollary 5.41, we start by recalling the two constructions of the surgery exact triangle, and reformulating them in the language of chain complexes of attaching circles. The result then follows quickly from the pairing theorem.

5.10.1. The bordered construction of the surgery triangle. Let \mathcal{T} be a punctured genus one surface. Fix three linear curves β^∞ , β^{-1} , and β^0 in T of slopes ∞ , -1 , and 0 respectively. Let $\xi = \beta^\infty \cap \beta^{-1}$ and $\eta = \beta^{-1} \cap \beta^0$. Letting $\mathbb{I} = \{\infty, -1, 0\}$, we can view these curves and intersection points as determining an \mathbb{I} -filtered chain complex of bordered attaching curves, with $\eta^{\infty < -1} = \xi$, $\eta^{-1 < 0} = \eta$, and $\eta^{\infty < 0} = 0$. (This is in fact Example 3.17 with a little reparameterization. The reader bothered by the ordering $\infty < -1$ should think of ∞ as $-\infty$.)

Choose α to consist of two curves going out to the puncture in \mathcal{T} , one parallel to β^∞ and the other parallel to β^0 . Place the basepoint z and order the chords at the punctures as illustrated in Figure 11.

The associated \mathbb{I} -filtered type D module $\widehat{CFD}(\alpha, \{\beta^i\}_{i \in \mathbb{I}}, \{\eta^{i < j}\}_{i, j \in \mathbb{I}}, z)$ has three summands:

$$\widehat{CFD}(\mathcal{H}_\infty) = \widehat{CFD}(\alpha, \beta^\infty, z) \quad \widehat{CFD}(\mathcal{H}_{-1}) = \widehat{CFD}(\alpha, \beta^{-1}, z) \quad \widehat{CFD}(\mathcal{H}_0) = \widehat{CFD}(\alpha, \beta^0, z).$$

These have generating sets $\{r\}$, $\{a, b\}$, and $\{n\}$ in filtration levels ∞ , -1 , and 0 respectively, and differentials

$$\begin{aligned}
 \delta^1 r &= \rho_{23} \otimes r & \delta^1 a &= (\rho_1 + \rho_3) \otimes b \\
 \delta^1 b &= 0 & \delta^1 n &= \rho_{12} \otimes n.
 \end{aligned}$$

The maps changing filtration are:

$$\begin{aligned} F^{\infty < -1}(r) &= (\rho_2 \otimes a) + b & F^{-1 < 0}(a) &= n \\ F^{-1 < 0}(b) &= \rho_2 \otimes n & F^{\infty < 0} &= 0. \end{aligned}$$

All these maps can be computed by counting holomorphic disks in the torus. Note that $F^{-\infty < -1}$ and $F^{-1 < 0}$ are the maps denoted φ and ψ in [LOT08, Section 11.2].

We can also consider a third map $\theta = F^{0 < \infty}$, the map gotten by reordering $\{-1, 0, \infty\}$, i.e., counting holomorphic triangles based at the intersection point $\zeta = \beta^\infty \cap \beta^0$. This map can be readily computed as

$$(5.42) \quad \theta(n) = (\rho_1 + \rho_3) \otimes r.$$

There is an explicit isomorphism

$$(5.43) \quad \widehat{CFD}(\mathcal{H}_{-1}) \xrightarrow{\simeq} \text{Cone}(\theta: \widehat{CFD}(\mathcal{H}_0) \rightarrow \widehat{CFD}(\mathcal{H}_\infty))$$

given by

$$\begin{aligned} b &\mapsto r + \rho_2 \otimes n \\ a &\mapsto n. \end{aligned}$$

(Isomorphisms

$$\begin{aligned} \widehat{CFD}(\mathcal{H}_0) &\simeq \text{Cone}(\phi: \widehat{CFD}(\mathcal{H}_\infty) \rightarrow \widehat{CFD}(\mathcal{H}_{-1})) \\ \widehat{CFD}(\mathcal{H}_\infty) &\simeq \text{Cone}(\psi: \widehat{CFD}(\mathcal{H}_{-1}) \rightarrow \widehat{CFD}(\mathcal{H}_0)) \end{aligned}$$

are even easier to write down.)

Now, suppose that Y is a 3-manifold with a framed knot (K, λ) in it and let \mathcal{H} be a bordered diagram for $Y \setminus \text{nbdd}(K)$. The identification

$$\text{Cone}(\theta: \widehat{CFD}(\mathcal{H}_0) \rightarrow \widehat{CFD}(\mathcal{H}_\infty)) \simeq \widehat{CFD}(\mathcal{H}_{-1})$$

can be tensored with $\widehat{CFA}(\mathcal{H})$ to give a quasi-isomorphism

$$(5.44) \quad \widehat{CFA}(\mathcal{H}) \boxtimes \widehat{CFD}(\mathcal{H}_{-1}) \simeq \text{Cone}(\mathbb{I}_{\widehat{CFA}(\mathcal{H})} \boxtimes \theta: \widehat{CFA}(\mathcal{H}) \boxtimes \widehat{CFD}(\mathcal{H}_0) \rightarrow \widehat{CFA}(\mathcal{H}) \boxtimes \widehat{CFD}(\mathcal{H}_\infty)).$$

The pairing theorem for bigons [LOT08, Theorem 1.3] gives quasi-isomorphisms

$$(5.45) \quad \begin{aligned} \widehat{CFA}(\mathcal{H}) \boxtimes \widehat{CFD}(\mathcal{H}_\infty) &\simeq \widehat{CF}(Y) \\ \widehat{CFA}(\mathcal{H}) \boxtimes \widehat{CFD}(\mathcal{H}_{-1}) &\simeq \widehat{CF}(Y_{-1}(K)) \\ \widehat{CFA}(\mathcal{H}) \boxtimes \widehat{CFD}(\mathcal{H}_0) &\simeq \widehat{CF}(Y_0(K)). \end{aligned}$$

Together, Equations (5.44) and (5.45) rise to a long exact sequence relating $\widehat{HF}(Y_0(K))$, $\widehat{HF}(Y)$, and $\widehat{HF}(Y_{-1}(K))$ as in Equation (5.37).

As noted in [LOT14a], this approach extends easily to give surgery triangles for bordered Floer homology as well. Suppose Y is a bordered 3-manifold with two boundary components and (K, λ) is a framed knot in the interior of Y . Choose an arced bordered Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ with three boundary components for $Y \setminus \text{nbdd}(K)$. (Here, “arced” means, for instance, that there is a component of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ adjacent to all three components of $\partial\Sigma$, and we place the basepoint z in this region.) There are bordered trimodules $\widehat{CFDDD}(\mathcal{H})$, $\widehat{CFDDA}(\mathcal{H})$ and so on associated to \mathcal{H} , one for each labeling of the boundary components

by elements of $\{D, A\}$; the definitions of these trimodules are trivial adaptations of the definitions of the bimodules in [LOT15]. Consider in particular the trimodules $\widehat{CFDDA}(\mathcal{H})$, $\widehat{CFDAA}(\mathcal{H})$ and $\widehat{CFAAA}(\mathcal{H})$ where the boundary component corresponding to K is labeled by A . The pairing theorem gives quasi-isomorphisms

$$(5.46) \quad \begin{aligned} \widehat{CFDDA}(\mathcal{H}) \boxtimes \widehat{CFD}(\mathcal{H}_\infty) &\simeq \widehat{CFDD}(Y) \\ \widehat{CFDDA}(\mathcal{H}) \boxtimes \widehat{CFD}(\mathcal{H}_{-1}) &\simeq \widehat{CFDD}(Y_{-1}(K)) \\ \widehat{CFDDA}(\mathcal{H}) \boxtimes \widehat{CFD}(\mathcal{H}_0) &\simeq \widehat{CFDD}(Y_0(K)), \end{aligned}$$

and similarly with DD replaced by DA or AA . Combining Equations (5.44) (and its DA and AA versions) and Equation (5.46) gives

$$(5.47) \quad \begin{aligned} \widehat{CFDD}(Y_{-1}(K)) &\simeq \text{Cone}(\theta_{LOT}: \widehat{CFDD}(Y_0(K)) \rightarrow \widehat{CFDD}(Y_\infty(K))) \\ \widehat{CFDA}(Y_{-1}(K)) &\simeq \text{Cone}(\theta_{LOT}: \widehat{CFDA}(Y_0(K)) \rightarrow \widehat{CFDA}(Y_\infty(K))) \\ \widehat{CFAA}(Y_{-1}(K)) &\simeq \text{Cone}(\theta_{LOT}: \widehat{CFAA}(Y_0(K)) \rightarrow \widehat{CFAA}(Y_\infty(K))), \end{aligned}$$

the most natural analogues of the exact triangle (5.37) in this more complicated algebraic setting. In particular, the map $\theta = \theta_{LOT}$ from Formula (5.39) is defined as:

$$\begin{aligned} \theta_{LOT} = \mathbb{I} \boxtimes \theta: \widehat{CFDD}(Y_0(K)) &\simeq \widehat{CFDDA}(\mathcal{H}) \boxtimes \widehat{CFD}(\mathcal{H}_0) \\ &\rightarrow \widehat{CFDDA}(\mathcal{H}) \boxtimes \widehat{CFD}(\mathcal{H}_\infty) \simeq \widehat{CFDD}(Y_\infty(K)). \end{aligned}$$

Of course, there are also surgery triangles for bordered 3-manifolds with connected boundary, via the same argument but with trimodules replaced by bimodules.

5.10.2. *The original construction of the surgery triangle.* The original construction of the surgery exact triangle is somewhat different. Again, suppose that Y is a three-manifold with a framed knot (K, λ) in it. There is a corresponding Heegaard triple as in [OSz04a] $(\Sigma, \alpha, \gamma, \beta, z)$ representing the two-handle cobordism from the surgery $Y_\lambda(K)$ to Y . In particular, $(\Sigma, \gamma, \beta, z)$ represents a connected sum of $(S^2 \times S^1)$'s. Counting triangles with one input a cycle Θ representing the top-graded homology class in $\widehat{HF}(\Sigma, \gamma, \beta, z)$ gives a map

$$m_2(\cdot \otimes \Theta_{\gamma, \beta}): \widehat{CF}(\alpha, \gamma, z) \rightarrow \widehat{CF}(\alpha, \beta, z).$$

In the language of chain complexes of attaching circles, take a bordered diagram $\mathcal{H} = (\Sigma_1, \alpha^1, \beta^1)$ for the complement of K in Y . Consider the Heegaard triple $(\mathcal{T}, \alpha, \{\beta^0, \beta^\infty\}, z)$ described above, where we think of $(\mathcal{T}, \{\beta^0, \beta^\infty\}, z)$ as a two-step chain complex of attaching circles, using the cycle $\eta^{0 < \infty} = \beta^0 \cap \beta^\infty$. We can glue this to β^1 (in the sense of Definition 3.40), which we now think of as a one-step complex of attaching circles in Σ_1 , to get a two-step chain complex

$$(\Sigma^1, \{\beta^1\}, z) \# (\mathcal{T}, \{\beta^0, \beta^\infty\}, z) = (\Sigma, \{\gamma, \beta\}, z),$$

equipped with distinguished chain $\Theta_{\gamma, \beta} \in \widehat{CF}(\Sigma_1 \# \Sigma, \gamma, \beta, z)$. Pairing this two-step complex with $\alpha^1 \cup \alpha$ gives the two-step complex which is the mapping cone of

$$m_2(\cdot \otimes \Theta_{\gamma, \beta}): \widehat{CF}(\Sigma_1 \cup \Sigma, \alpha^1 \cup \alpha, \gamma, z) \rightarrow \widehat{CF}(\Sigma_1 \cup \Sigma, \alpha^1 \cup \alpha, \beta, z).$$

The proof of the surgery exact triangle from [OSz04a] gives a quasi-isomorphism

$$\widehat{CF}(\mathcal{H} \cup \mathcal{H}_{-1}) \simeq \text{Cone}(m_2(\cdot, \Theta_{\gamma, \beta}): \widehat{CF}(\Sigma, \alpha, \gamma, z) \rightarrow \widehat{CF}(\Sigma, \alpha, \beta, z)).$$

The long exact sequence (5.37) relating $\widehat{HF}(Y_0)$, $\widehat{HF}(Y)$, and $\widehat{HF}(Y_{-1})$ follows at once.

5.10.3. *The two constructions agree.*

Proof of Proposition 5.40. It is immediate from the definitions that the Heegaard triple $(\Sigma, \alpha^1, \beta^1) \# (\mathcal{T}, \alpha, \{\beta^0, \beta^\infty\}, z)$ agrees with the Heegaard triple $(\Sigma, \alpha, \gamma, \beta, z)$ associated to the two-handle cobordism from Y_0 to Y . Further, the chain $\Theta_{\gamma, \beta}$ constructed in the gluing of chain complexes represents the top-graded generator of $\widehat{HF}(\gamma, \beta, z)$. So, the commutative square is a direct consequence of the pairing theorem for triangles, Proposition 5.35. \square

Proof of Corollary 5.41. This is the special case of Proposition 5.40 in which both of the boundary components of Y are empty, together with the observation that θ_{OS} is given by $m_2(\Theta_{\gamma, \beta}, \cdot)$. \square

6. IDENTIFYING THE SPECTRAL SEQUENCES

6.1. The multi-diagram for the branched double cover. Let L be a link in S^3 . In this section we describe a particular Heegaard multi-diagram for the branched double cover of the cube of resolutions of L . Our main reason for interest in this diagram is that it decomposes as a concatenation of particularly simple bordered Heegaard diagrams, but it has other nice properties, as well. In fact, this diagram is essentially the one studied by J. Greene [Gre13]. (See Proposition 6.4 for a precise statement.)

Draw the plat closure of a $2n$ -braid. Rotate it 90° clockwise (so that the maxima are on the right and the minima are on the left, rather than top and bottom), as this convention is best adapted to the bordered setting. The knot projection is thought of as gotten from a diagram for the unlink with n maxima (on the right) and n minima (on the left) by modifying the picture so as to introduce crossings between various consecutive strands, as specified by the braid. Decompose the knot projection into three regions: the *cap region*, consisting of the n maxima, the *cup region*, consisting of the n minima, and the *braid region*, which contains the braid. We will assume the following properties of the knot projection:

- It is given as a standard plat closure of a braid (i.e., where all the cups (respectively caps) happen at the same time).
- In the braid, the first two strands never cross each other – i.e., the first strand is stationary throughout the braid. (We will think of the “first two strands” as the top two strands in the picture.)

Both of these properties can be at the cost of possibly introducing more crossings.

We describe a corresponding Heegaard diagram for the branched double cover of the n -component unlink, and then describe local changes at the crossings needed to obtain the desired multi-diagram associated to the projection of L . (For an example of the resulting multi-diagram, see Figure 13.)

Label the strands in the (trivial) braid from bottom to top s_1, \dots, s_{2n} . We think of these as $2n$ horizontal segments in the unlink projection. We first build the part of a Heegaard multi-diagram associated to the braid region. This part of the Heegaard diagram is built from an annulus, thought as a rectangle in the plane whose top and bottom edges are identified. The annulus is equipped with horizontal arcs which will eventually be used to build closed

curves (α -circles) in the Heegaard diagram, as follows. Each strand s_i for $1 < i < 2n - 1$ is replaced by a pair of horizontal arcs, one just above and one just below the horizontal strand s_i , while the strands s_1 and s_{2n-1} both induce single horizontal arcs in the Heegaard diagram (and s_{2n} induces none). For $1 < i < 2n - 1$, label the arc in the Heegaard diagram just above s_i by $(s_{i-1}, s_i)_+$, and the one just below s_i by $(s_i, s_{i+1})_-$, replacing s_1 by $(s_1, s_2)_-$ and s_{2n-1} by $(s_{2n-2}, s_{2n-1})_+$. Thus, coming up from the bottom, the horizontal arcs in the Heegaard diagram are labeled as follows:

$$\begin{aligned} & (s_1, s_2)_-, (s_2, s_3)_-, (s_1, s_2)_+, \\ & (s_3, s_4)_-, (s_2, s_3)_+, (s_4, s_5)_-, (s_3, s_4)_+, \\ & \dots \\ & (s_i, s_{i+1})_-, (s_{i-1}, s_i)_+, (s_{i+1}, s_{i+2})_-, (s_i, s_{i+1})_+ \\ & \dots \\ & (s_{2n-2}, s_{2n-1})_+. \end{aligned}$$

So far, we have specified the Heegaard diagram in the braid region, provided that there are no crossings. We will next describe the cap and cup regions of the diagram, and finally we turn to the modifications to the braid region needed in the case where there are crossings.

At the cap region, we add $(n - 1)$ one-handles, after which we close off the pairs of arcs $(s_i, s_{i+1})_-$ and $(s_i, s_{i+1})_+$ for $i = 1, \dots, 2n - 1$, and draw $(n - 1)$ β -circles. In more detail, at the cap region (the right of the diagram) we close off the rightmost endpoints of the arcs $(s_{2i-1}, s_{2i})_-$ and $(s_{2i-1}, s_{2i})_+$ for $i = 1, \dots, n$, by arcs denoted $(s_{2i-1}, s_{2i})_r$. Next, we draw n circles labeled β_i^r for $i = 1, \dots, n - 1$, where the i^{th} β -circle encircles the rightmost endpoints of $(s_{2i-2}, s_{2i-1})_+$ and $(s_{2i}, s_{2i+1})_-$, except when $i = 1$, in which case it encircles only $(s_1, s_2)_-$. We draw these circles small enough that they are disjoint from the arcs $(s_{2i-1}, s_{2i})_r$. Next, we stabilize the picture by attaching one-handles joining up the endpoints of $(s_{2i}, s_{2i+1})_-$ and $(s_{2i}, s_{2i+1})_+$; call the resulting arc $(s_{2i}, s_{2i+1})_r$. Now the arcs $(s_i, s_{i+1})_-$ and $(s_i, s_{i+1})_+$ are connected in the cap region for $i = 1, \dots, 2n - 2$.

The diagram in the cup region is the mirror image of the diagram in the cap region. We label the $n - 1$ new β -circles here $\{\beta_i^\ell\}_{i=1}^{n-1}$, and the arcs joining $(s_i, s_{i+1})_-$ and $(s_i, s_{i+1})_+$ in the left region by $(s_i, s_{i+1})_\ell$.

So far, we have a surface of genus $2n - 2$, equipped with $2n - 2$ β -circles. For $i = 1, \dots, 2n - 2$, the closed curves $(s_i, s_{i+1})_\ell \cup (s_i, s_{i+1})_- \cup (s_i, s_{i+1})_+ \cup (s_i, s_{i+1})_r$ are our α -circles. This gives a Heegaard diagram for $\#_{i=1}^{2n-2}(S^2 \times S^1)$, which is the branched double cover of the plat closure of the trivial braid on $2n$ strands.

We describe now how to modify the Heegaard diagram (in the braid region), in the presence of crossings. If the k^{th} crossing occurs between the strands s_i and s_{i+1} , then choose small disks D_k^- intersecting $(s_i, s_{i+1})_-$ and D_k^+ intersecting $(s_i, s_{i+1})_+$. Remove the interiors of these disks and identify their boundaries via reflection across a horizontal axis. This has the effect of attaching a one-handle with two arcs running through it to the Heegaard multi-diagram. With this one-handle attachment, we have increased the number of α -circles by one. There will be four choices for how to add a corresponding β -circle. Either we take a meridian for the newly attached one-handle, μ_k ; or we take a β -circle λ_k which runs through the handle, meeting only the two arcs $(s_{i-1}, s_i)_+$ and $(s_i, s_{i+1})_-$; or we take a curve which is one of the two resolutions of $\mu_k \cup \lambda_k$. For the negative braid generator, we let β_k^0 be the meridian μ_k , β_k^1 be the longitude λ_k , and β_k^∞ be their resolution pictured on the top of

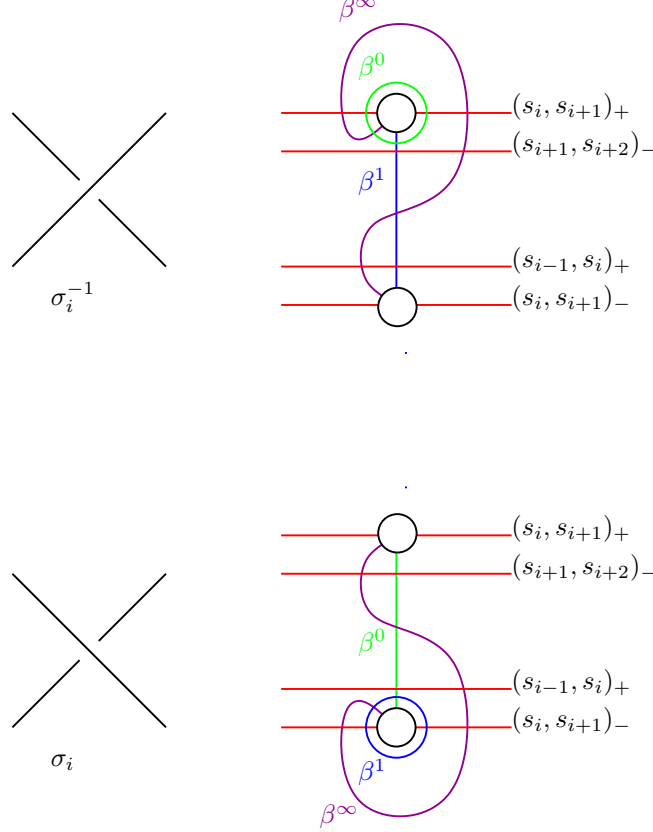


FIGURE 12. **Modifications for the Heegaard multi-diagram at each crossing.** The two types of braid generator are illustrated in the left column. This corresponds to a modification of the Heegaard multi-diagram as illustrated on the right.

Figure 12. For the positive braid generator, we let β_k^0 be the longitude λ_k , β_k^1 be the meridian μ_k , and β_k^∞ be the other resolution, pictured on the bottom of Figure 12.

We will see in Lemma 6.1 that the Heegaard diagram $(\Sigma, \alpha, \beta^\infty, z)$ represents the branched double cover $\Sigma(L)$ of L .

We will think of the diagram $(\Sigma, \alpha, \{\beta^j\}_{j \in \{0,1,\infty\}}, z)$ as sliced up into $c+2$ slices: reading right to left, the first of these corresponds to the (rightmost) cap region, c of these correspond to the crossings and the last corresponds to the cup regions. In more detail, we cut up the Heegaard multi-diagram along $c+1$ concentric circles S_0, \dots, S_c based at some central point off to the right of the diagram. The 0^{th} slice of the Heegaard multi-diagram is the region encircled by S_0 (which is drawn large enough to contain all of the cap region); for $k = 1, \dots, c$, the k^{th} slice of the Heegaard multi-diagram is the region in the annulus between the circle S_{k-1} and S_k . The $(c+1)^{\text{st}}$ slice is the region outside S_c (including a point at infinity). The crossing modifications are arranged so that both D_k^- and D_k^+ occur in the k^{th} slice (in the same order as the crossings appear in the knot projection).

The resulting surface has genus $g = 2n - 2 + c$, and it is equipped with a $(2n - 2 + c)$ -tuple of attaching circles α . Moreover, for each $j \in \{0, 1, \infty\}^c$, there is also a $(2n - 2 + c)$ -tuple of curves β^j , consisting of all the above β -circles on the right and the left of the diagram, and

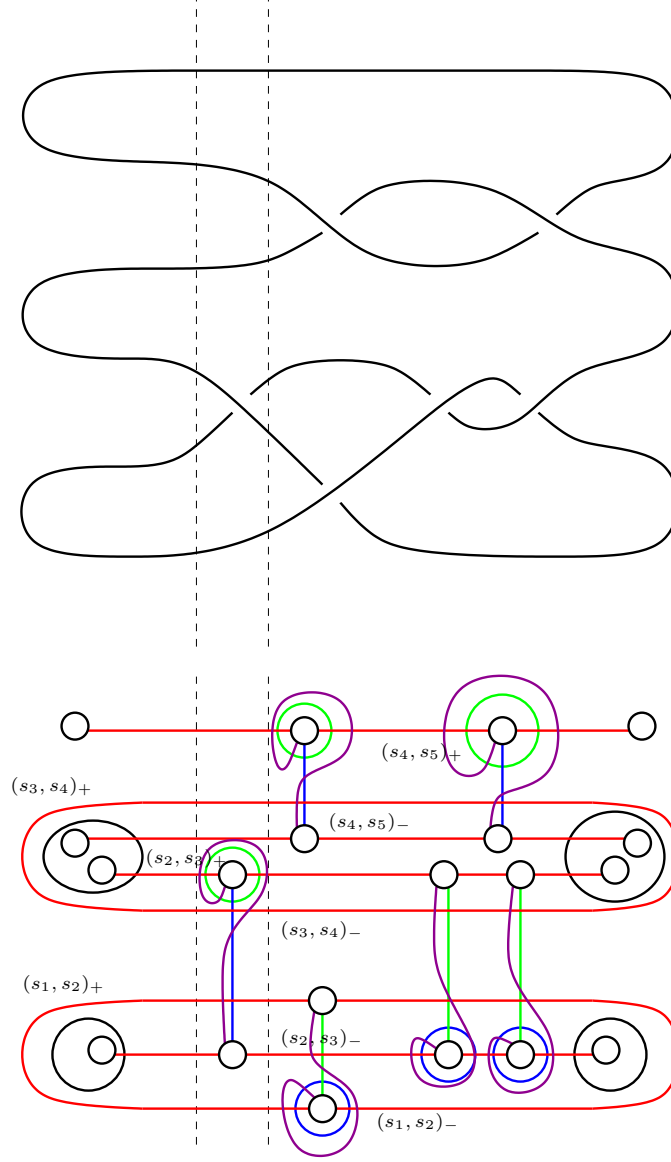


FIGURE 13. **Heegaard multi-diagram for a branched double cover.** Start from the knot projection above, and build the diagram below. We have distinguished a vertical slice of the knot diagram containing a single crossing. The corresponding vertical slice of the Heegaard multi-diagram, after suitable stabilizations, is a bordered multi-diagram for a Dehn twist.

choosing β_k^0 , β_k^1 , or β_k^∞ at the k^{th} crossing as prescribed by the component j_k . An example is illustrated in Figure 13.

This diagram is not ideal for our purposes: the slices of this diagram are not yet bordered multi-diagrams in the sense of Definition 4.31, as most of the α -arcs run from one boundary component to the other. Specifically, the k^{th} slice, involving a crossing between strands s_i and s_{i+1} , has pairs of arcs corresponding to $(s_\ell, s_{\ell+1})_-$ and $(s_\ell, s_{\ell+1})_+$ for all $\ell \neq i$ which connect the two boundary components. We can, however, attach one-handles to reconnect these pairs of arcs so that each of the new arcs has both endpoints on one boundary component, as in

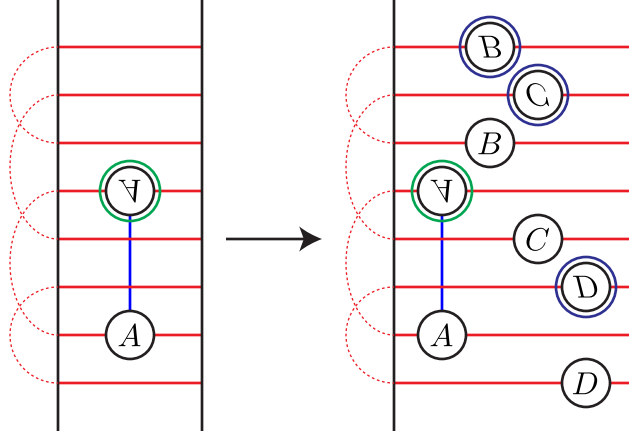


FIGURE 14. **Attaching handles and reconnecting arcs.** Each letter corresponds to a handle.

Figure 14. This modification has the effect of increasing the genus by another $c(2n - 3)$, and introducing the same number of new α -circles and β -circles β^j (which are the same for all choices of $j \in \{0, 1, \infty\}^c$). This additional stabilization does not increase the total number of Heegaard Floer generators for any of the diagrams – it serves only to make sensible the holomorphic curve counts in the various bordered slices. We refer to the destabilized version as the *small diagram for the branched double cover*; but we will typically consider its stabilized version.

Finally, the Heegaard multi-diagram is equipped with a basepoint corresponding to the point at infinity. This can be extended to an arc of basepoints crossing all the slices, but disjoint from all attaching circles, in a straightforward way.

For $k = 0, \dots, c + 1$, write the part of the Heegaard multi-diagram in the k^{th} slice as $(\Sigma_k, \alpha_k, \{\beta_k^j\}_{j \in \{0, 1, \infty\}}, z)$, so that $k = 0$ and $k = c + 1$ correspond to the cup and cap regions.

If $Y = \Sigma(L)$ is the branched double cover of a link L with c crossings, then Y is equipped with a framed link L' whose components are in one-to-one correspondence with the crossings of L . (Each crossing c in the diagram for L specifies an arc whose boundary lies in L . The branched double cover of this arc gives the link L' . The framing of this link is specified so that 0-framed surgery gives the branched double cover of the braid-like or the anti-braid-like resolution, depending on whether the crossing is of type σ_i or σ_i^{-1}).

Lemma 6.1. *The $\mathbb{I} = \{0, 1, \infty\}^c$ -filtered chain complex of attaching circles*

$$(\Sigma, \{\beta^j\}_{j \in \{0, 1, \infty\}^c}, z) = \#_{k=0}^{c+1} (\Sigma_k, \{\beta_k^j\}_{j \in \{0, 1, \infty\}}, z)$$

is a chain complex of attaching circles associated to the framing changes as in Definition 3.60 on the link $L' \subset Y$ specified by α and $\{\beta_k^j\}$.

It is natural to see this from the bordered perspective; so we postpone the proof a moment.

6.2. The bordered decomposition of the multi-diagram. The slices of the diagram considered above were studied in [LOT14a]. As noted above, each slice is a bordered Heegaard (multi-)diagram with two boundary components. Each boundary component is parameterized by the *linear pointed matched circle*, as in [LOT14a, Section 5] or Figure 15, which



FIGURE 15. **The linear pointed matched circle.** The genus 3 (i.e., $n = 4$) case is shown.

we think of as the 2-sphere branched at $2n$ collinear points. Each of the matched pairs in the linear pointed matched circle \mathcal{Z} specifies a circle in the surface $F(\mathcal{Z})$ containing the core of the corresponding handle. We call these the *generating curves* in $F(\mathcal{Z})$. These generating curves correspond to branched double covers of straight arcs connecting two consecutive of the $2n$ collinear points in the 2-sphere.

Proof of Lemma 6.1. We find this easiest to verify one slice at a time.

Consider the k^{th} slice and suppose for definiteness that the crossing there is a positive braid generator. By inspection, the intersection of α and β^1 with this slice specifies the identity cobordism, which we think of as the branched double cover of the sphere times an interval branched at $2n$ non-crossing arcs connecting the two boundary components (i.e., the braid-like resolution of the positive crossing). We also claim that the intersection of α and β^0 with this slice specifies the branched double cover of a cup followed by a cap (i.e., the anti-braid-like resolution of the positive crossing). This can be seen by noting that β_k^0 is a copy of one of the generating curves on F , supported at the mid-level of the product cobordism. It follows that α and β^0 represents some (Morse) surgery on the knot in the branched double cover of the trivial braid on $2n$ strands, which is the branched double cover on an arc connecting two of the consecutive strands. To see that it is, in fact, a cup followed by a cap (which is surgery with respect to the surface framing) follows from straightforward homological considerations. The positive Dehn twist (branched double cover of the positive braid generator) is now obtained as surgery on this same curve with a new framing (-1 with respect to the surface framing): it is specified as a suitable resolution of the sum of β_k^0 and β_k^1 . Permuting the roles of the ambient and the surgered three-manifold, we have that β_k^0 and β_k^1 denote two different framings on the knot in the three-manifold Y corresponding to the k^{th} crossing.

The bottom and top pieces correspond to standard handlebodies, thought of as branched double covers of the 3-ball branched along n arcs.

Gluing the pieces together, the result follows. \square

Lemma 6.2. *The bordered diagram $(\Sigma_k, \alpha_k, \beta_k^\infty, \mathbf{z})$ represents a Dehn twist along one of the preferred generating curves. Moreover, counting holomorphic triangles in the two-step chain complex $(\Sigma_k, \alpha_k, \{\beta_k^j\}_{j \in \{0,1\}}, \theta^{0 < 1})$ gives a map*

$$F^- : \widehat{CFDA}(\mathbb{I}) \rightarrow \widehat{CFDA}(\check{\sigma}_i) \text{ or } F^+ : \widehat{CFDA}(\check{\sigma}_i) \rightarrow \widehat{CFDA}(\mathbb{I}),$$

according to whether the k^{th} crossing is of the form σ_i or σ_i^{-1} respectively, whose mapping cone is identified with $\widehat{CFDA}(\Sigma_i, \alpha_i, \beta_i^\infty, z)$.

Proof. The first sentence follows from an inspection of the diagram. The identification between the mapping cone of the triangle map and the Dehn twist follows from the bordered proof of the surgery exact triangle [LOT14a, Theorem 5] together with Proposition 5.40, which shows the maps in the surgery exact triangle come from counting holomorphic triangles. \square

Lemma 6.3. *The map from Lemma 6.2 induced by counting holomorphic triangles in the bordered diagram $(\Sigma_k, \alpha_k, \{\beta_k^0, \beta_k^1\}, z)$,*

$$F^-: \widehat{CFDA}(\mathbb{I}) \rightarrow \widehat{CFDA}(\check{\sigma}_i) \text{ or } F^+: \widehat{CFDA}(\check{\sigma}_i) \rightarrow \widehat{CFDA}(\mathbb{I}),$$

agrees up to homotopy with the maps (with the same notation) from [LOT14a].

Proof. To keep the notation simple, we will talk about F^- ; the proof for F^+ is the same.

By [LOT14a, Propositions 7.11 and 7.26], for the type DD maps we have

$$F_{DD, \text{hol}}^- = F_{DD, \text{comb}}^-: \widehat{CFDD}(\mathbb{I}) \rightarrow \widehat{CFDD}(\check{\sigma}_i).$$

The map $F_{\text{comb}}^-: \widehat{CFDA}(\mathbb{I}) \rightarrow \widehat{CFDA}(\check{\sigma}_i)$ from [LOT14a] is (by definition) gotten by tensoring $F_{DD, \text{comb}}^-$ with the identity map of $\widehat{CFAA}(\mathbb{I})$:

$$\begin{array}{ccc} \widehat{CFDA}(\mathbb{I}) & \xrightarrow{F_{\text{comb}}^-} & \widehat{CFDA}(\check{\sigma}_i) \\ \simeq \downarrow & & \uparrow \simeq \\ \widehat{CFDD}(\mathbb{I}) \boxtimes \widehat{CFAA}(\mathbb{I}) & \xrightarrow{F_{DD, \text{comb}}^- \boxtimes \mathbb{I}} & \widehat{CFDD}(\check{\sigma}_i) \boxtimes \widehat{CFAA}(\mathbb{I}). \end{array}$$

To see that $F_{\text{comb}}^- \sim F_{\text{hol}}^-$, tensor both sides with the identity map of $\widehat{CFDD}(\mathbb{I})$. It follows from the pairing theorem (Theorem 7) that $F_{\text{hol}}^- \boxtimes \mathbb{I}_{DD} \sim F_{DD, \text{hol}}^-$, and it follows from the fact that $\widehat{CFDD}(\mathbb{I}) \boxtimes \widehat{CFAA}(\mathbb{I}) \simeq \widehat{CFDA}(\mathbb{I}) = [\mathbb{I}]$ that $F_{\text{comb}}^- \boxtimes \mathbb{I}_{DD} \sim F_{DD, \text{comb}}^-$. But tensoring with $\widehat{CFDD}(\mathbb{I})$ is a quasi-equivalence of dg categories, so this implies that $F_{\text{hol}}^- \sim F_{\text{comb}}^-$, as desired. \square

6.3. Putting together the pieces.

Proof of Theorem 2. Using the diagram explained above, Lemma 6.1 identifies the chain complex of attaching circles used to construct the spectral sequence from [OSz05] with the chain complex of attaching circles which decomposes as a concatenation of bordered diagrams for Dehn twists in the linear pointed matched circle.

The pairing theorem for polygons (Theorem 8) then identifies the filtered complex with the one gotten by an iterated tensor product of DA bimodule morphisms, where the morphisms are defined by counting pseudo-holomorphic triangles.

Lemma 6.3 then identifies these DA bimodule morphisms with the combinatorially defined morphisms constructed in [LOT14a], and the filtered complex gotten from the iterated tensor product of these combinatorial models induces the spectral sequence from [LOT08]. \square

6.4. Kauffman states and Greene's diagram. As mentioned earlier, the Heegaard multi-diagram considered here is a stabilization of the diagram considered by Greene [Gre13]. As such, if we forget about the filtration on our chain complex, we end up with a chain complex for the branched double cover whose generators correspond to Kauffman states. This correspondence can be seen locally. The β -circle at each crossing meets four of the

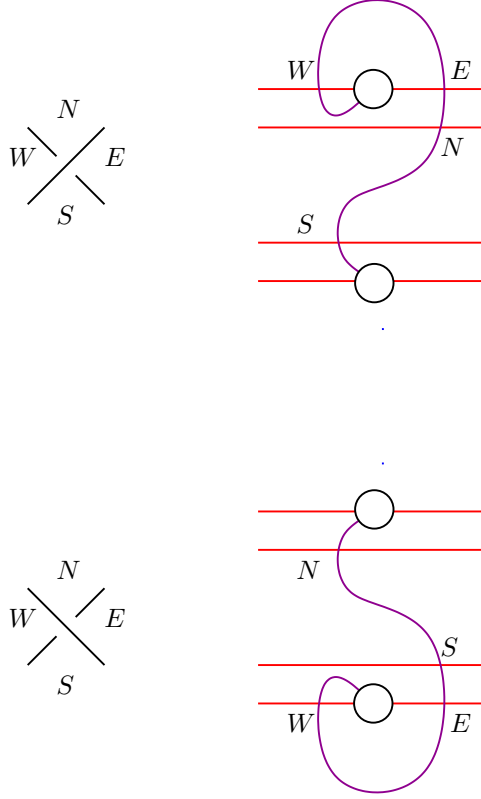


FIGURE 16. **Local correspondence between Kauffman states and Heegaard Floer generators.**

α -curves (except if the strand is one of the two extremal strands). Thus, each generator picks out one of these four intersection points. Which of those four is chosen corresponds to a local choice of Kauffman states, as indicated in Figure 16. It is easy to see that these local correspondences piece together to give a correspondence between Heegaard Floer generators and Kauffman states (compare [Gre13]; Proposition 6.4; and also [OSz03]). See Figure 17 for a global example.

This correspondence extends to the resolutions: at each resolution, there is a correspondence between Kauffman states in the resolved diagram and Heegaard-Floer generators. It is interesting to note that for the complete resolutions, the Heegaard multi-diagram is typically not admissible: a disconnected, complete resolution has no Kauffman states, but the Floer homology is non-trivial.

Our aim now is to relate our Heegaard diagrams with Greene's. As a first step, we paraphrase Greene's description.

Let K be a projection of a knot, with one marked edge. For consistency with Section 6.1 we assume (unlike Greene) that it is the plat closure of a braid, lying on its side, with no crossings involving the top strand. We think of the top strand as the marked edge.

Take a regular neighborhood T of the knot diagram in the plane of the knot projection, with the induced orientation from the plane. This region T will eventually form half the Heegaard surface (the “top half”). The boundary of T is a collection of circles. We label all of those circles except the top edge as α -circles.

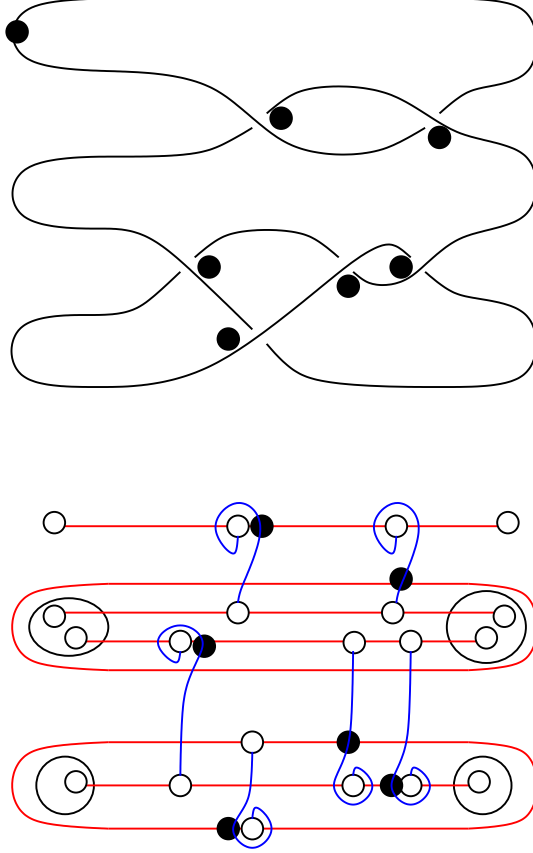


FIGURE 17. **A knot, a Kauffman state, the branched double cover Heegaard multi-diagram, and corresponding Heegaard Floer generator.** For the Kauffman state indicated above, we have drawn components of the corresponding Heegaard Floer generator. The Heegaard Floer generator has more components which are not indicated; but those are all uniquely determined.

We now construct portions of β -circles corresponding to crossings. Think of crossings as involving two strands, one of which connects SW and NE , and the other of which connects SE and NW . Correspondingly, there are four corners in ∂T at each crossing, which we label N , S , E , and W . We draw pairs of arcs in T , which will eventually close up to form β -circles corresponding to the crossings. If the SW/NE strand is an overcrossing, one arc connects W to N and the other S to E ; otherwise, one arc connects E to N and the other connects S to W .

There is one additional β arc which cuts across the marked edge in T .

The Heegaard surface now is gotten by doubling T along its boundary. Let B denote the other half of the double (the “bottom half”). In the bottom half, there is a β arc which closes up the special β arc at the marked edge. Also, at each crossing, we draw pairs of β arcs in B , consisting of an arc connecting N to S and another connecting E to W . These arcs are all drawn to be pairwise disjoint, but are not contained in a neighborhood of the crossing. (This can be done uniquely, up to diffeomorphism.)

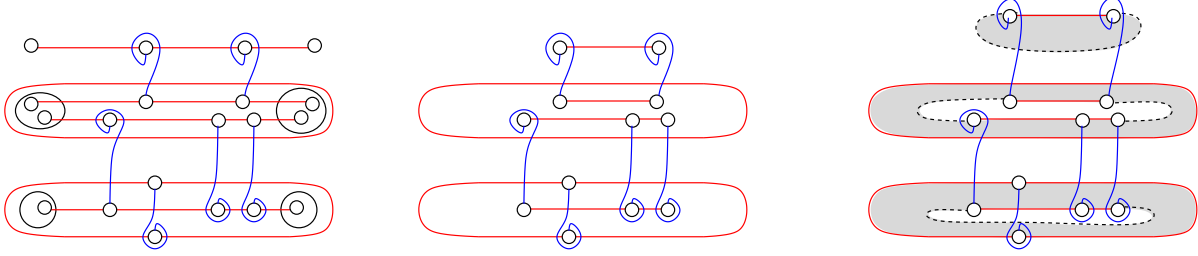


FIGURE 18. **Modifications to the small standard diagram to get Greene's diagram.** Passing from the first to the second picture is a sequence of destabilizations at the cup and cap regions. The third picture is equipped with the curve γ (drawn dashed), and the region T' is shaded.

This is, essentially, Greene's description. Note that the distinguished β -circle corresponding to the marked edge meets a single α -circle; thus, this pair of curves can be destabilized to construct what we shall call the *small Greene diagram*.

A knot can be rotated 180° around the x -axis, to obtain a new planar projection. If the knot is the plat closure of a braid, this has the effect of replacing each σ_i with σ_{n-i} . We will call the resulting knot diagram the *rotated diagram*.

Proposition 6.4. *For any connected plat braid diagram for K , the standard small Heegaard diagram for the branched double cover of K (as in this paper) can be destabilized $2n - 2$ times (supported in the cup and cap regions) so that it becomes homeomorphic to the small Greene diagram associated to the rotated diagram of K . In particular, there is a canonical identification between generators for the standard diagram studied here and Greene's diagram (for the rotate of K), which identifies absolute gradings of generators.*

Proof. Consider the standard small diagram for the branched double cover of a knot. As a first step, erase all the β -circles supported inside cup and cap regions and the α -circles which meet those β -circles. Destabilize all the handles supported in the cup and cap regions. This is the destabilization described at the beginning of the proposition; call this diagram \mathcal{D} .

It is perhaps easiest to see the identification after finding regions corresponding to T and B in \mathcal{D} . To this end, we will describe a circle γ in \mathcal{D} .

Draw a portion of γ which is mostly parallel, but below, the topmost horizontal arcs, and then enters the handles at the leftmost and rightmost ends of the topmost arcs. Next, continue γ so that it consists of arcs which connect the two handles which are the leftmost ends of the arcs $(s_i, s_{i+1})_\ell$, and similarly so that it consists of arcs which connect the two handles which are the rightmost ends of the arcs $(s_i, s_{i+1})_r$ (all this for $i = 1, \dots, 2n - 2$). Finally, run an arc nearly parallel and just below $(s_2, s_3)^-$, running through the leftmost and rightmost handles at the two ends of this curve.

All the α curves together with γ divide our Heegaard surface into two regions, one of which we denote T' and the other B' . (We label these so that the point at infinity is contained in B' .) See Figure 18.

It is now straightforward to see that T' corresponds to T in Greene's diagram: this follows from local considerations at each crossing as in Figure 19. Note that in the homeomorphism between T and T' , the directions N and S are reversed, while E and W are not. Locally in T (and in T') there are four directions which can be connected to the other local pieces associated to the crossings. Label these directions NE , NW , SE , and SW in the obvious

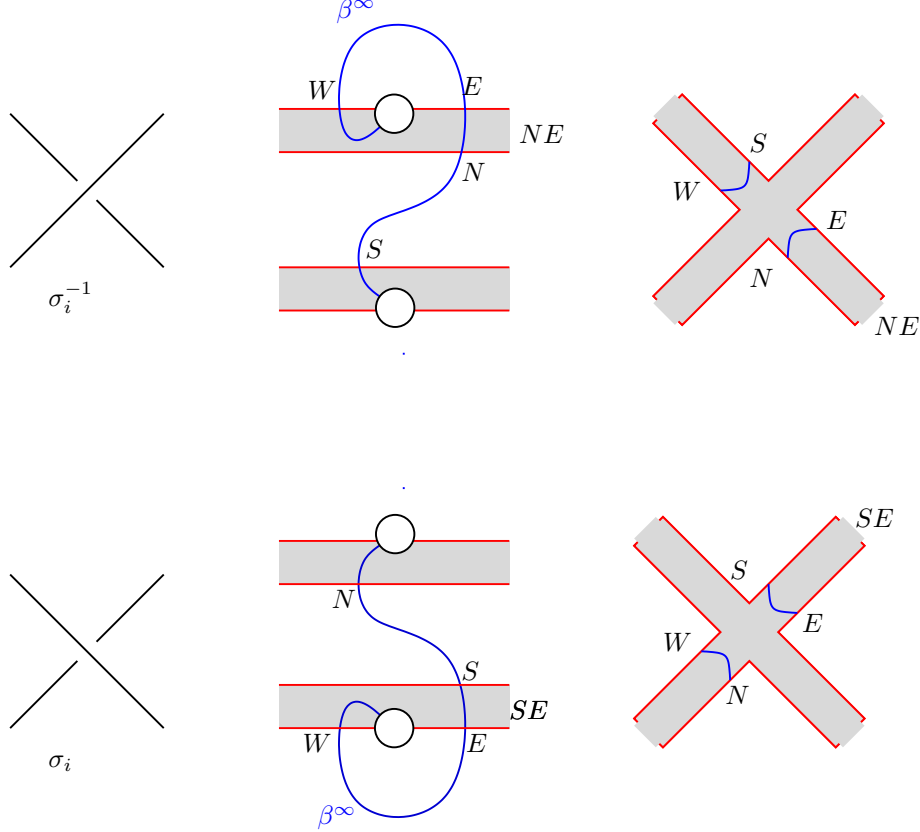


FIGURE 19. **Local identification between the small standard diagram and Greene's diagram.** At the left is a crossing; in the middle the corresponding portion of the standard diagram (with shaded T'), and at right the corresponding portion of Greene's diagram (with T shaded).

way. The homeomorphism between T and T' also switches NE and SE , and NW and SW . This effectively switches σ_i to σ_{n-i} . This homeomorphism clearly extends over B and B' . \square

Proposition 6.4 is particularly useful because of the thoroughness of Greene's work: for example, he explicitly computes the gradings of the generators on the branched double cover, and these computations extend quickly to compute the gradings of the branched double cover described here. Moreover, the proof of Proposition 6.4 above can be adapted easily to give a comparison between our multi-diagram and Greene's multi-diagram.

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